

Math 424, Spring 2024
Lecture 01 - History & Motivation

This course is an introduction to a branch of mathematics called "complex analysis" which is ultimately concerned with the study of special types of functions which are called "holomorphic." As we will see, these functions are almost magical, satisfying several very nice properties, and a large amount of theory has developed around holomorphic functions and related ideas.

Although interesting in its own right, complex analysis (aka the study of holomorphic functions) has proven itself to be an indispensable tool in modern mathematics and areas relying heavily on sophisticated mathematical machinery, such as quantum mechanics.

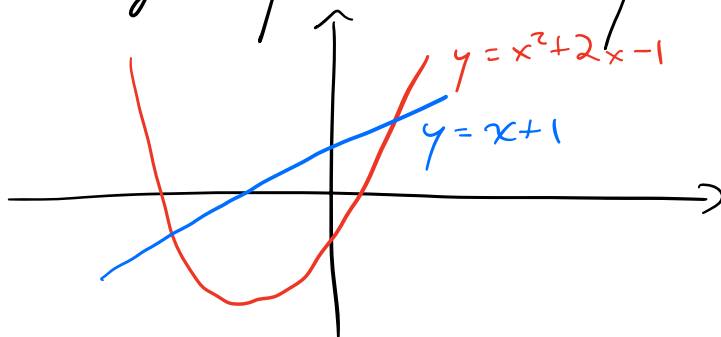
Before jumping into holomorphic functions and all of the beautiful theory surrounding them, though, we need to spend some time "setting the stage," which here means we need to get comfortable with complex numbers. Over the next few lectures we'll introduce complex numbers, see the arithmetic and geometric properties of these numbers, and work our way up to introducing mappings between

complex numbers. First, though, let's start with some background and history.

The "complex numbers" we'll soon describe are numbers that arise in considering roots of polynomial equations. In particular, these quantities allow us to make sense of square roots of negative numbers, such as $\sqrt{-1}$. But why would we ever care about roots of negative numbers? How could such things be remotely useful?

To see the utility of considering roots of a negative number, let's consider a very concrete geometric problem: Suppose you wanted to determine the point of intersection between a line (the set of points in the plane $\mathbb{R}^2$ satisfying some equation $ax+by=c$ — or, if you insist, $y=mx+b$) and a quadratic, some eqn $y=\alpha x^2+\beta x+\gamma$.

For example, if we want to find the intersection of $y=x+1$ and $y=x^2+2x-1$



Then we are lead to the equation

$$x+1 = x^2 + 2x - 1$$

which we may rewrite as

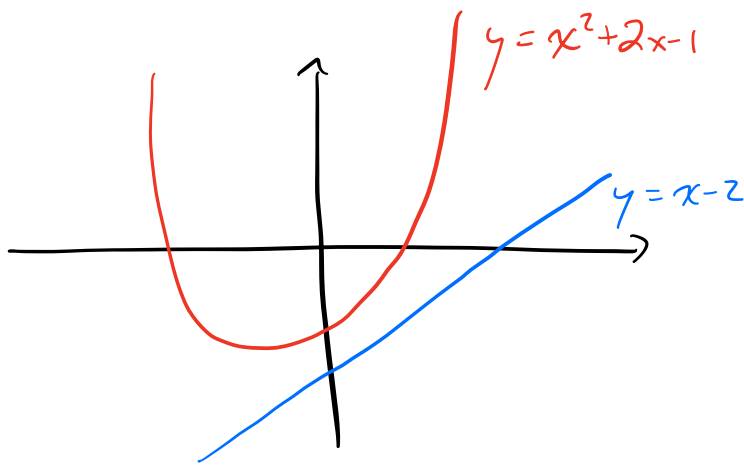
$$x^2 + x - 2 = 0$$

We can easily solve this equation using the quadratic formula to find

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-2)}}{2} = \frac{-1 \pm \sqrt{9}}{2} \\ &= \frac{-1 \pm 3}{2} = \frac{-1+3}{2} \text{ or } \frac{-1-3}{2} \\ &= 1, -2 \end{aligned}$$

So the x -coordinates of our intersections are $x=1$ and $x=-2$, and we can plug these back into either $y=x+1$ or $y=x^2+2x-1$ to get the corresponding y -coordinates, and see our curves intersect at $(-2, -1)$ and $(1, 2)$.

But what if our curves were instead $y=x^2+2x-1$ and $y=x-2$?



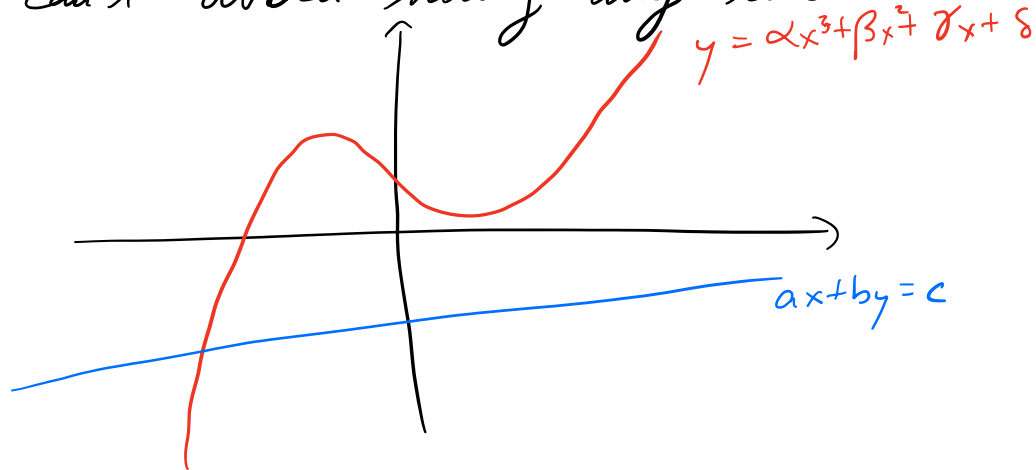
These curves no longer intersect and if we tried to algebraically determine their intersection, we'd have

$$\begin{aligned}
 x^2 + 2x - 1 &= x - 2 \\
 \Rightarrow x^2 + x + 1 &= 0 \\
 \Rightarrow x &= \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot 1}}{2} = \frac{-1 \pm \sqrt{-3}}{2}
 \end{aligned}$$

Notice we have a square root of a negative that appears. Mathematicians living before the "dawn" of complex numbers viewed square roots of negatives as impossibilities and viewed the equation above as simply meaning the curves never intersected, and were fine to leave it at that.

But we could also consider intersecting a line $ax + by = c$ with a cubic curve $y = \alpha x^3 + \beta x^2 + \gamma x + \delta$. Geometrically we can see such curves will always intersect — the cubic always has a

side approaching $-\infty$ and a side approaching ∞ ,
so it can't avoid hitting any line



Of course finding our points of intersection will boil down to finding the roots of some cubic polynomial: after writing each equation as $y = \text{"blah blah blah"}$, we'd equate our two right-hand side expressions, then combine like-terms and have something of the form

$$Ax^3 + Bx^2 + Cx + D = 0$$

If we're lucky this will factor, but for a "generic" choice of coefficients A, B, C, D this won't factor in any nice/obvious way. It'd be nice, then, if we had something like the quadratic formula but for cubic equations. Luckily such a thing does exist. We'll present the result we need now, and leave its derivation as an exercise.

We begin with a helpful definition and lemma:

A cubic polynomial is called depressed if its quadratic term is zero. That is, a depressed cubic is one of the form $Ax^3 + Bx + C$.

Lemma

By changing coordinates, every cubic may be transformed into a depressed cubic.

Pf
(Homework exercise.)

□

What's convenient about depressed cubics is that we have a nice expression for their roots.

Lemma (Cardano's formula)

The depressed cubic equation

$$x^3 - px - q = 0$$

is solved by

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}$$

Pf
(Homework exercise.)

□

Notice that if $\left(\frac{q}{2}\right)^2 < \left(\frac{p}{3}\right)^3$, then this expression

⑥

will necessarily involve square roots of negatives. Keeping in mind that we were lead to this expression by trying to determine where a line and a cubic intersect, we can't just ignore roots of negatives as being indicative of there being no intersection points, as we have noticed a line and cubic will always intersect.

The Italian mathematician Rafael Bombelli considered this oddity in the 1500's, while thinking of the solutions to the equation

$$x^3 = 15x + 4$$

(Which we could interpret as giving the points of intersection between the cubic $y = x^3$ and the line $y = 15x + 4$.)

From simple inspection, Bombelli noted $x=4$ solved this equation ($4^3=64$, and $15 \cdot 4 + 4 = 60 + 4 = 64$). However, Cardano's formula above doesn't directly give us 4. Instead, writing the equation as

$$x^3 - 15x - 4 = 0$$

and so $p = 15$, $q = 4$ in Cardano's formula gives

$$\begin{aligned}
x &= \sqrt[3]{\frac{9}{2} + \sqrt{\left(\frac{9}{2}\right)^2 - \left(\frac{1}{3}\right)^3}} + \sqrt[3]{\frac{9}{2} - \sqrt{\left(\frac{9}{2}\right)^2 - \left(\frac{1}{3}\right)^3}} \\
&= \sqrt[3]{\frac{4}{2} + \sqrt{\left(\frac{4}{2}\right)^2 - \left(\frac{15}{3}\right)^3}} + \sqrt[3]{\frac{4}{2} - \sqrt{\left(\frac{4}{2}\right)^2 - \left(\frac{15}{3}\right)^3}} \\
&= \sqrt[3]{2 + \sqrt{2^2 - 5^3}} + \sqrt[3]{2 - \sqrt{2^2 - 5^3}} \\
&= \sqrt[3]{2 + \sqrt{4 - 125}} + \sqrt[3]{2 - \sqrt{4 - 125}} \\
&= \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \\
&= \sqrt[3]{2 + \sqrt{121 \cdot (-1)}} + \sqrt[3]{2 - \sqrt{121 \cdot (-1)}} \\
&= \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}
\end{aligned}$$

Now we have a conundrum: on the one hand, we know $x=4$ should be a solution to our equation, but on the other hand Cardano's formula tells us the solution should be this "nonsensical" expression involving square roots of negatives,

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}$$

Bombelli was bold enough to suppose that perhaps these quantities were equal—maybe

there's some reasonable way to show

$$\sqrt[3]{2+11\sqrt{-1}} + \sqrt[3]{2-11\sqrt{-1}} = 4$$

Here Bombelli makes a very bold assumption — he goes so far as to suppose $\sqrt[3]{2+11\sqrt{-1}}$ could be written as $2+a\sqrt{-1}$, and also that $\sqrt[3]{2-11\sqrt{-1}}$ could be written as $2-a\sqrt{-1}$ for some a . If that were true, the $a\sqrt{-1}$ terms would cancel out giving us back 4 when added together. Now let's see if we can justify Bombelli's "wild thought."

If $\sqrt[3]{2+11\sqrt{-1}} = 2+a\sqrt{-1}$, then $(2+a\sqrt{-1})^3 = 2+11\sqrt{-1}$.
Now we compute

$$\begin{aligned}(2+a\sqrt{-1})^3 &= 2^3 + 3 \cdot 2^2 \cdot a\sqrt{-1} + 3 \cdot 2 \cdot (a\sqrt{-1})^2 + (a\sqrt{-1})^3 \\&= 8 + 12a\sqrt{-1} + 6a^2 \cdot (-1) + a^3(-1)\sqrt{-1} \\&= 8 - 6a^2 + (12a - a^3)\sqrt{-1}\end{aligned}$$

If this were to equal $2+11\sqrt{-1}$, then

$$\begin{aligned}8 - 6a^2 &= 2 \\12a - a^3 &= 11\end{aligned}$$

Notice we can solve the first equation for a

to obtain

$$\begin{aligned}8 - 6a^2 &= 2 \\ \Rightarrow 6 &= 6a^2 \\ \Rightarrow 1 &= a^2 \\ \Rightarrow a &= \pm 1\end{aligned}$$

If we plug these into our second equation we have

$$\begin{aligned}12 \cdot 1 - 1 \\ = 12 - 1 \\ = 11\end{aligned}$$

$$\begin{aligned}12(-1) - (-1)^3 \\ = -12 + 1 \\ = -11\end{aligned}$$

Thus $a=1$, and $\sqrt[3]{2+11\sqrt{-1}} = 2+\sqrt{-1}$.

We can similarly show $(2-\sqrt{-1})^3 = 2-11\sqrt{-1}$,
so $\sqrt[3]{2-11\sqrt{-1}} = 2-\sqrt{-1}$.

Using these computations, we see that Cardano's formula does in fact give us 4!

$$\begin{aligned}\sqrt[3]{2+11\sqrt{-1}} + \sqrt[3]{2-11\sqrt{-1}} &= 2+\sqrt{-1} + 2-\sqrt{-1} \\ &= 4\end{aligned}$$

So, these "non-sensical" square roots of negative give us real answers to problems we may

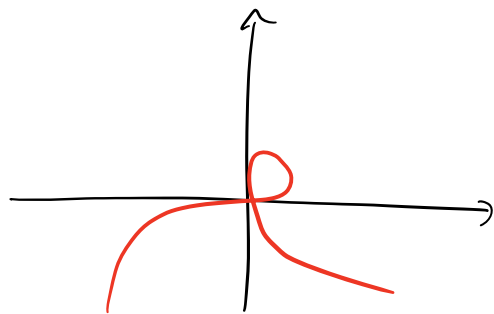
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care about!

Today, complex numbers arise and are used in several diverse areas of math: geometry, number theory, and algebra all use complex numbers in fundamental ways. And, as paradoxical as it may sound at first, many problems in math become easier when complex numbers are considered.

As one example, whose details are outside the scope of this class, let's briefly mention an important result from algebraic geometry. Suppose you had two curves in the plane which were given by polynomial equations - so one curve is given by some equation say

$$x^4 + xy^2 - 3xy + y^3 = 0$$

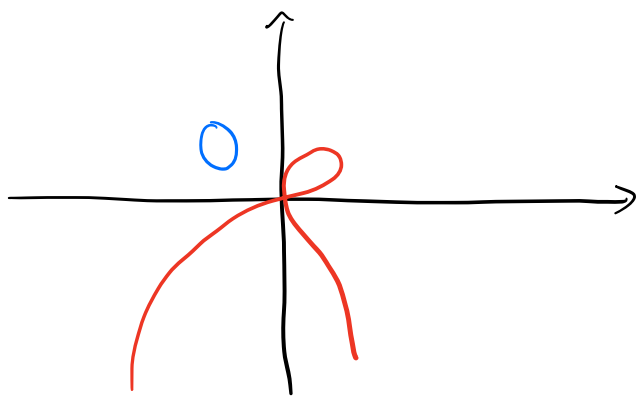


and we have a second curve, maybe some kind of quadratic in x and y ,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

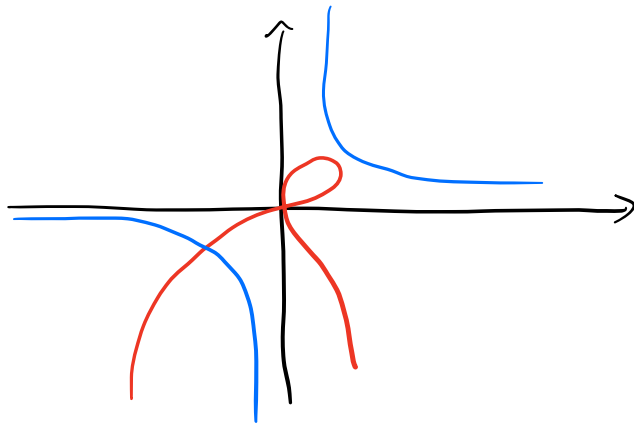
Of course, different choices of $A, B, \dots, F$ give us

different curves, and we may ask how many times our quartic above intersects the quadratic. In the "normal" real plane this number of intersections changes based on $A, B, \dots, F$



$$A=1, B=1, C=2, D=4, E=0, F=4$$

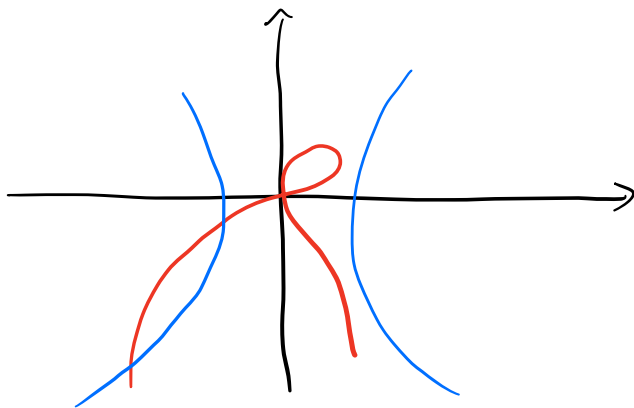
No intersections



$$A=C=D=E=0$$

$$B=-2 \quad F=10$$

One intersection



$$A=-4, B=1, C=2, D=8, E=0, F=4$$

Two intersections

When we consider the complex version of this problem, though (ie., we allow x and y to be complex numbers), an important result called Bézout's theorem says there

will always intersect 8 times! More generally, curves given by polynomials of degrees m and n always intersect mn times when we consider the complex (x,y) -points satisfying the polynomial equation. (There are several details here we're sweeping under the rug, but morally Bézout's theorem says what we've stated.)

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Lecture 02 - The Arithmetic & Geometry of the Complex Plane

In the last lecture we saw some history of how mathematicians we lead to seriously consider square roots of negatives. Now we jump forward a couple of hundred years and give a "modern" introduction to the complex numbers.

We begin by giving a special name to the square root of -1 : we call $\sqrt{-1}$ the imaginary unit and denote this quantity by i . Note $i^2 = -1$.

A purely imaginary number is a real multiple of i — that is, an expression of the form iy where $y \in \mathbb{R}$.

A complex number is the sum of a real number x and a purely imaginary number iy , $x + iy$. The set of all complex numbers is denoted $\mathbb{C}$:

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$$

We'll often let the variable z denote an arbitrary complex number, $z = x + iy$.

Given a complex number $z = x + iy$, we call x

the real part of the number and denote it $\text{Re}(z)$.

We call y (just the real number y , not iy) the imaginary part of the number and denote it $\text{Im}(z)$.

For example, if $z = 7 + 3i$, then $\text{Re}(z) = 7$, $\text{Im}(z) = 3$;

if $z = \pi - 5i$, then $\text{Re}(z) = \pi$, $\text{Im}(z) = -5$;

if $z = 4i$, then $\text{Re}(z) = 0$, $\text{Im}(z) = 4$

if $z = 13$, then $\text{Re}(z) = 13$, $\text{Im}(z) = 0$.

Extending the last example, we can think of each real number as being a complex number, just one that has zero imaginary part. Thus we can interpret the set of real numbers $\mathbb{R}$ as a subset of the complex numbers $\mathbb{C}$: $\mathbb{R} \subseteq \mathbb{C}$.

Given two complex numbers, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we can combine them together to get a new complex number in a few different ways

- The sum $z_1 + z_2$ is defined as

$$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2) = (\text{Re}(z_1) + \text{Re}(z_2)) + i(\text{Im}(z_1) + \text{Im}(z_2))$$

- The product $z_1 \cdot z_2$ is defined as

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1) \cdot (x_2 + iy_2) \\ &= x_1 x_2 + x_1 i y_2 + i y_1 x_2 + i^2 y_1 y_2 \end{aligned}$$

$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

With these definitions of addition and multiplication, we can show that the set of complex numbers forms a "field." In general, a field is a set F with two binary operations, $+$ (addition) and $\cdot$ (multiplication) so that several axioms are satisfied:

- Addition is associative: $\forall a, b, c \in F, (a+b)+c = a+(b+c)$
- Addition is commutative: $\forall a, b \in F, a+b = b+a$
- There is an additive identity: $\exists 0 \in F \forall a \in F, 0+a = a$
- There are additive inverses: $\forall a \in F, \exists (-a) \in F, a+(-a) = 0$
- Multiplication is associative: $\forall a, b, c \in F, (a \cdot b) \cdot c = a \cdot (b \cdot c)$
- Multiplication is commutative: $\forall a, b \in F, a \cdot b = b \cdot a$
- There is a multiplicative identity: $\exists 1 \in F \forall a \in F, 1 \cdot a = a$
- Every non-zero element has a multiplicative inverse:
 $\forall a \in F \setminus \{0\}, \exists a^{-1} \in F, a \cdot a^{-1} = 1$
- Multiplication distributes across addition:
 $\forall a, b, c \in F, a \cdot (b+c) = a \cdot b + a \cdot c$

The reals $\mathbb{R}$ and rationals $\mathbb{Q}$ are also examples of fields, but the integers $\mathbb{Z}$ do not form a field.

The take-away from having a field is that you can do arithmetic and all the normal rules you expect to hold will hold. For example, you can prove that a field never has "zero divisors;" i.e.,

the product of any two non-zero elements is not zero. This isn't true in other more general contexts. E.g., in the set of 2×2 matrices we can have non-zero matrices that multiply together to give the zero matrix, such as $\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Since we claim (without proof) that $\mathbb{C}$ is a field, we are required to have multiplicative inverses. I.e., we need to be able to divide complex numbers. Let's see if we can work through what $\frac{1}{z}$ must be. That is given $z = x + iy$ can we determine the real and imaginary parts of $\frac{1}{z}$? If we could, then we could divide any number by z .

For example, it will turn out that

$$\frac{1}{3+4i} = \frac{3}{25} - \frac{4}{25}i$$

Once we know that, we could easily divide any complex number by $3+4i$. E.g.,

$$\begin{aligned} \frac{25+50i}{3+4i} &= (25+50i) \cdot \frac{1}{3+4i} \\ &= (25+50i) \cdot \left(\frac{3}{25} - \frac{4}{25}i \right) \\ &= 3 - 4i + 6i - 8i^2 \\ &= 11 + 2i \end{aligned}$$

Now, how can we determine $\frac{1}{z}$? One reasonable way to do this would be to suppose $z = x + iy$ and $\frac{1}{z} = a + ib$, write out $z \cdot \frac{1}{z} = 1$ in terms of a, b, x, y and solve the resulting system of linear equations. This will work, but is tedious. There's a better way, though in order to describe it we need to introduce a new operation.

Given any complex number $z = x + iy$, we define its conjugate, denoted $\bar{z}$, to be the complex number

$$\bar{z} = x - iy$$

E.g., if $z = 5 + 2i$, then $\bar{z} = 5 - 2i$.

Before seeing how the conjugate will be helpful for computing $\frac{1}{z}$, let's go ahead and mention some basic properties of conjugates.

Lemma

If $z_1, z_2 \in \mathbb{C}$, then $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$ and $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$.

Pf

We'll only prove that conjugates preserve sums, and leave the corresponding statement about products as an exercise. Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and consider

$$\begin{aligned}
\overline{z_1 + z_2} &= \overline{(x_1 + iy_1) + (x_2 + iy_2)} \\
&= \overline{(x_1 + x_2) + i(y_1 + y_2)} \\
&= (x_1 + x_2) - i(y_1 + y_2) \\
&= x_1 + x_2 - iy_1 - iy_2 \\
&= x_1 - iy_1 + x_2 - iy_2 \\
&= \overline{x_1 + iy_1} + \overline{x_2 + iy_2} \\
&= \overline{z_1} + \overline{z_2}
\end{aligned}$$

□

Another important property of conjugates, which we'll need in getting a nice expression for $\frac{1}{z}$, is that for every complex number z , the product of z with its conjugate is always real: $\forall z \in \mathbb{C}, z \cdot \bar{z} \in \mathbb{R}$.

Lemma

For every $z \in \mathbb{C}$, the product $z \cdot \bar{z}$ is real.

Pf

Let $z = x + iy$, so $\bar{z} = x - iy$. Then

$$\begin{aligned}
z \cdot \bar{z} &= (x + iy)(x - iy) = x^2 - ixy + ixy - i^2 y^2 \\
&= x^2 + y^2.
\end{aligned}$$

□

Let's now use the above to get a nice expression for $\frac{1}{z}$. Notice that regardless of what z is, $\frac{\bar{z}}{z} = 1$. Thus

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z \cdot \bar{z}} = \frac{x-iy}{x^2+y^2}.$$

So, for example, $\frac{1}{7+2i} = \frac{7-2i}{49+4} = \frac{7-2i}{53}$.

Now let's use this to divide two complex numbers. The general strategy is:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\bar{z}_2}{\bar{z}_2} = \frac{z_1 \cdot \bar{z}_2}{z_2 \cdot \bar{z}_2} = \frac{z_1 \cdot \bar{z}_2}{\text{Re}(z_2)^2 + \text{Im}(z_2)^2}$$

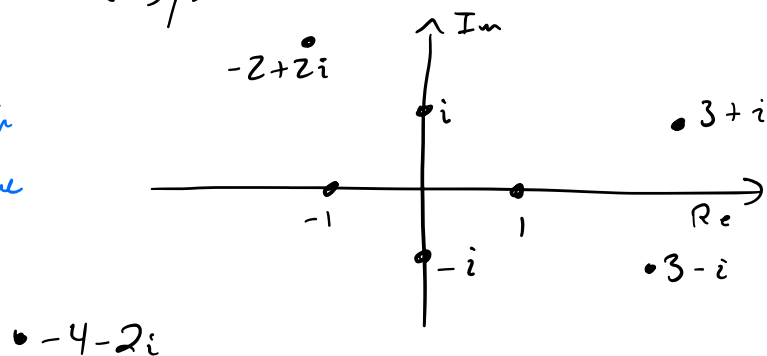
For example:

$$\begin{aligned} \frac{2+3i}{5+7i} &= \frac{2+3i}{5+7i} \cdot \frac{5-7i}{5-7i} = \frac{10-14i+15i-i^2 21}{5^2+7^2} = \frac{31+i}{25+49} \\ &= \frac{31}{74} + \frac{1}{74}i \end{aligned}$$

Now that we've described the basic arithmetic operations of the complex numbers, let's turn to their geometry.

First notice we can identify the set of complex numbers $\mathbb{C}$ with the Euclidean plane $\mathbb{R}^2$ in a natural way, identifying the number $x+iy$ with the point (x,y) .

Note $z=0$ is the origin of this plane

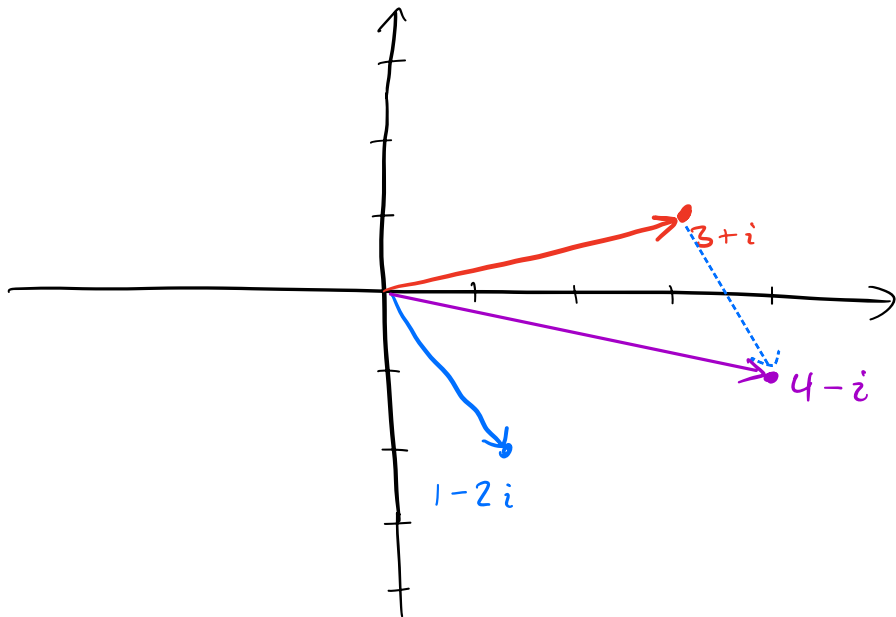


We often call the x -axis the real axis and the y -axis the imaginary axis in this context

Now, the addition of two complex numbers can be seen as the sum of the vectors in the plane from the origin to the corresponding point. For example, we can "algebraically" compute

$$(3+i) + (1-2i) = 4-i$$

by simply adding the real and imaginary parts of the numbers, but we can also visualize this using vectors. Recall that we perform vector addition in plane by placing the vectors "tip-to-tail" and filling in the missing side of the triangle:



Thinking of a complex number as corresponding to a point in the plane, or a vector from the origin to the point, there is a natural notion of a complex number's absolute value: it is simply

the length of the corresponding vector. To be a bit more explicit, if $z = x + iy$, we define $|z|$ to be $\sqrt{x^2 + y^2}$. So, for example,

$$\begin{aligned}|5 - 12i| &= \sqrt{5^2 + (-12)^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13\end{aligned}$$

Remark

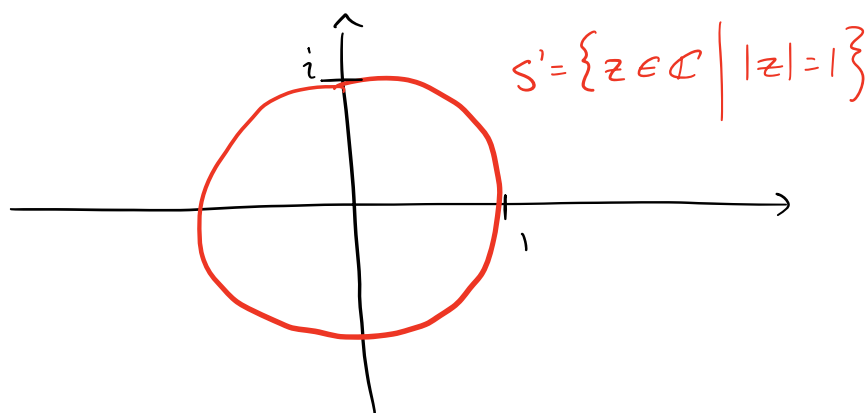
Notice

$$\begin{aligned}|z|^2 &= x^2 + y^2 \\ &= z \cdot \bar{z}\end{aligned}$$

Remark

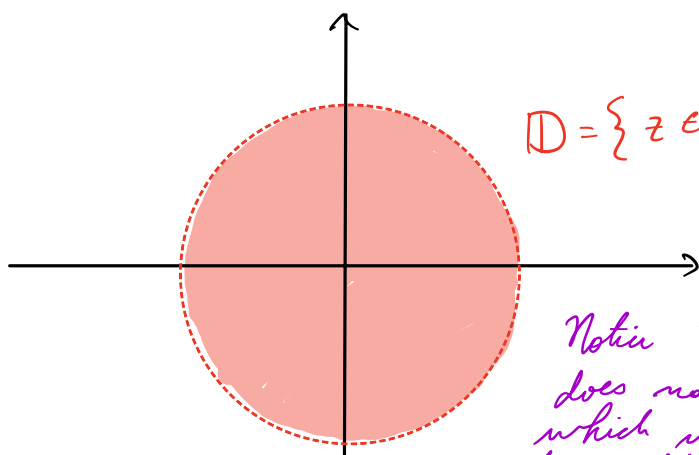
The absolute value $|z|$ is often called the modulus of z .

Notice that the unit circle in the plane could be described as the set of all $z \in \mathbb{C}$ satisfying $|z| = 1$. This set (the unit circle) is often denoted S^1 .



The unit disc, denoted $\mathbb{D}$, will play an important role in some more advanced applications of complex analysis (particularly hyperbolic geometry) and can be described

as all $z \in \mathbb{C}$ with modulus less than 1:

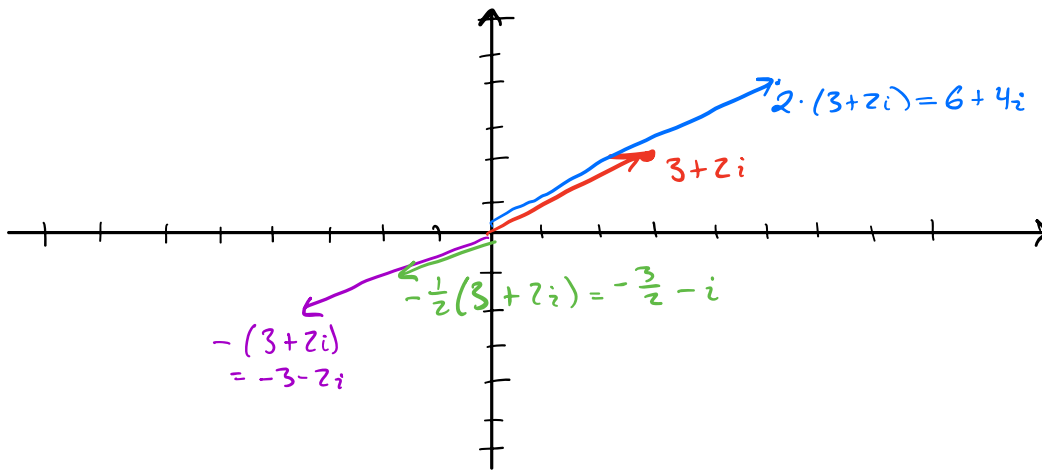


$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$$

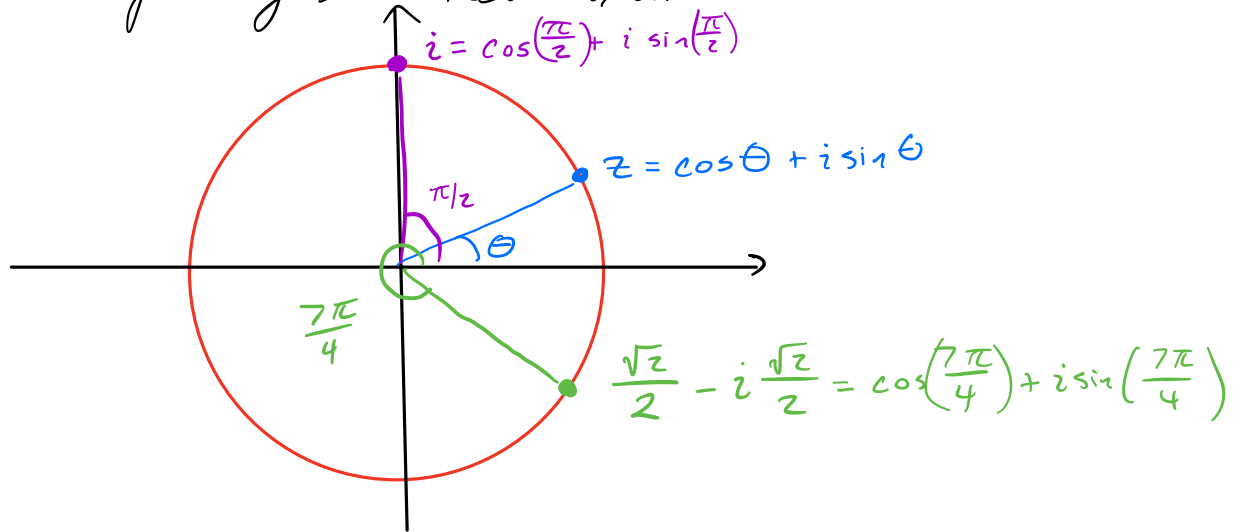
Notice $\mathbb{D} \not\ni \partial\mathbb{D}$ - the disc does not include its boundary, which we denote graphically by making the boundary dashed.

Products of complex numbers also have a nice geometric interpretation, but before describing it it will be helpful to see another way to describe complex numbers.

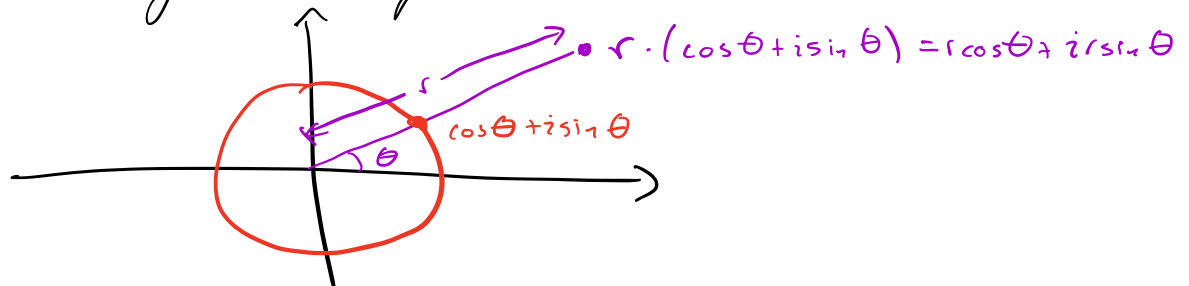
As a preliminary step to giving another description of complex numbers, let's first observe that if we multiply a complex number $z = x + iy$ by a real number $\lambda \in \mathbb{R}$, we have $\lambda z = \lambda x + i\lambda y$. When $\lambda > 0$, this simply stretches out our x and y coordinates by a factor of λ . If $\lambda < 0$, the point is rotated 180° about the origin, then stretched by $|\lambda|$. Thus multiplying a complex $z \in \mathbb{C}$ by a real $\lambda \in \mathbb{R}$ agrees with our usual notion of scalar multiplication of vectors in $\mathbb{R}^2$.



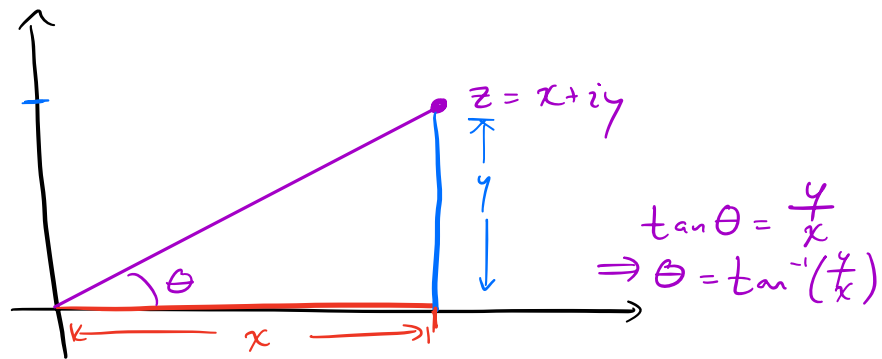
Now suppose z was a point on the unit circle. Then $z = \cos \theta + i \sin \theta$ where θ is the angle made by a segment from 0 to z , measured from the positive part of the real axis.



Now, if z was some point distance $r = |z|$ from 0 , then we can obtain z by taking a point $\cos \theta + i \sin \theta$ of S' , and multiplying it by r . (Here, θ is still the angle made by a segment from 0 to z , measured from the positive real axis.)



Notice some simple trig can tell us what θ is in terms of the real and imaginary parts of z :



And the corresponding r is just $r = |z| = \sqrt{x^2 + y^2}$

Of course, all we're really doing is describing the point in polar coordinates, just using complex notation.

But, just to recapitulate: the complex number $z = x + iy$ can be written as $z = r(\cos \theta + i \sin \theta)$ where $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(\frac{y}{x})$.

In this context, the angle θ above is often called the argument of z and is denoted $\arg(z)$.

Notice that $\arg(z)$ is not a function on z as there are infinitely-many possible values for $\arg(z)$, differing by multiples of 2π . For example $\arg(1+i)$ may be considered $\frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}, \dots$. So, $\arg(z)$ is not one number, but is a set of numbers. In order to have a function, we can limit ourselves to the angle in $(-\pi, \pi]$. This is the principal argument of z , denoted $\text{Arg}(z)$.

Let's see what this tells us about complex multiplication: suppose $z_1, z_2 \in \mathbb{C}$ where $z_1 = r(\cos \theta + i \sin \theta) = r \cos \theta + i r \sin \theta$, and $z_2 = \rho(\cos \omega + i \sin \omega) = \rho \cos \omega + i \rho \sin \omega$. Their product can be computed as

$$\begin{aligned} z_1 \cdot z_2 &= (r \cos \theta + i r \sin \theta) \cdot (\rho \cos \omega + i \rho \sin \omega) \\ &= r \rho \cos \theta \cos \omega - r \rho \sin \theta \sin \omega + i (r \rho \cos \theta \sin \omega + r \rho \sin \theta \cos \omega) \\ &= r \rho \left[(\cos \theta \cos \omega - \sin \theta \sin \omega) + i (\cos \theta \sin \omega + \sin \theta \cos \omega) \right] \end{aligned}$$

This looks pretty bad at first glance, but if we remember some trig identities, things simplify nicely. Recall that the angle-sum identities for sine and cosine are

$$\begin{aligned} \sin(\alpha + \beta) &= \cos \alpha \sin \beta + \sin \alpha \cos \beta \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \end{aligned}$$

Using these to simplify the above, we have

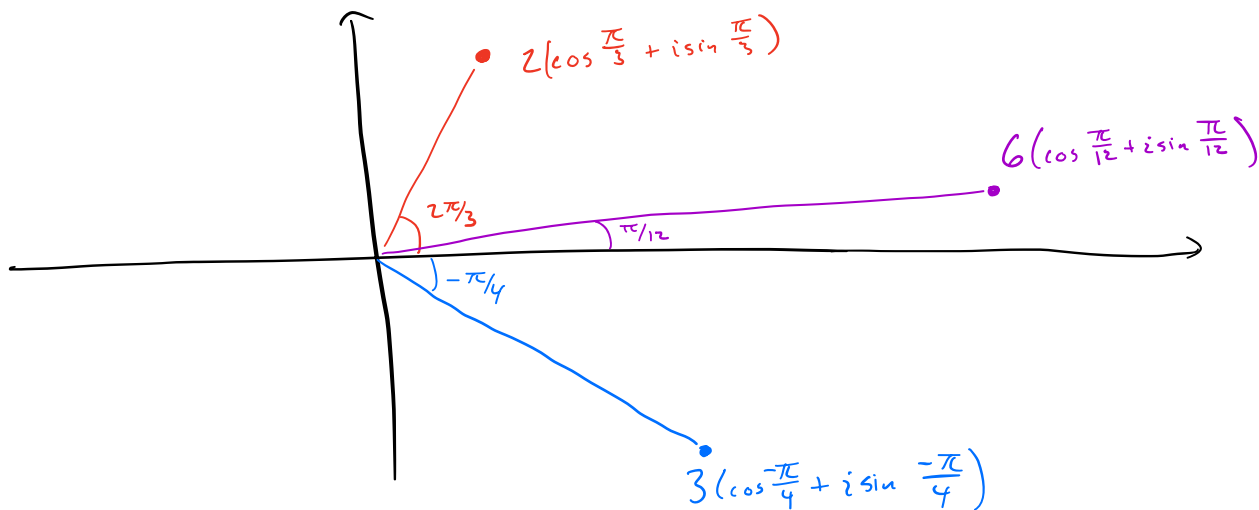
$$z_1 \cdot z_2 = r \rho (\cos(\theta + \omega) + i \sin(\theta + \omega))$$

Thus, the product of two complex numbers multiplies the moduli and adds the arguments!

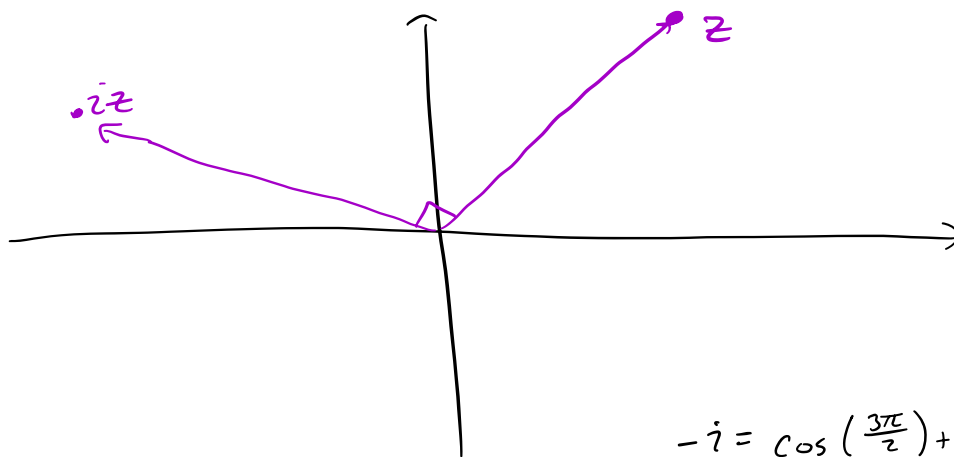
E.g.,

$$\frac{1}{3} - \frac{1}{4} = \frac{4-3}{12} = \frac{1}{12}$$

$$\begin{aligned} & \left(2 \cos\left(\frac{\pi}{3}\right) + i 2 \sin\left(\frac{\pi}{3}\right) \right) \cdot \left(3 \cos\left(\frac{-\pi}{4}\right) + i 3 \sin\left(\frac{-\pi}{4}\right) \right) \\ &= 6 \cos\left(\frac{\pi}{12}\right) + i 6 \sin\left(\frac{\pi}{12}\right) \end{aligned}$$



Note that multiplying z by i then rotates z by $\frac{\pi}{2}$ (90°) counterclockwise as $i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$



$$-i = \cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)$$

Multiplying by $-i$ corresponds to 90° rotation clockwise

Multiplying by a positive real is a 0° rotation ($\lambda = |\lambda|(\cos 0 + i \sin 0)$)

Multiplying by a negative real is a 180° rotation ($\lambda = |\lambda|(\cos \pi + i \sin \pi)$)

In general multiplication by $z \in S^1$ ($|z|=1$) corresponds to rotation by $\arg(z)$.

Remark

This interpretation of multiplying by a complex number z resulting in scaling by $|z|$ and rotating by $\arg(z)$ will be very important when we discuss complex mappings!

Our computation above basically proves the following lemma:

Lemma

For any complex numbers $z_1, z_2 \in \mathbb{C}$,

$$|z_1 z_2| = |z_1| \cdot |z_2| \quad \text{and} \quad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2).$$

□

Remark

One minor issue needs to be addressed here: if $\arg(z)$ isn't a single number, what does " $\arg(z_1) + \arg(z_2)$ " mean? We should really think of $\arg(z)$ as a set,

$$\arg(z) = \{ \operatorname{Arg}(z) + 2\pi m \mid m \in \mathbb{Z} \}$$

$$\begin{aligned} \text{E.g., } \arg(1+i) &= \left\{ \dots, \frac{-15\pi}{4}, \frac{-7\pi}{4}, \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}, \dots \right\} \\ &= \left\{ \frac{\pi}{4} + 2\pi m \mid m \in \mathbb{Z} \right\} \end{aligned}$$

If we have two sets of numbers, say $A, B \subseteq \mathbb{R}$ (or $A, B \subseteq \mathbb{C}$), we define the sum of two sets as the set of all possible sums of an element in A with an element in B :

$$A + B = \{ a + b \mid a \in A, b \in B \}$$

We'll finish up with one of the most commonly used inequalities in analysis:

Thm (The Triangle Inequality)

If $z_1, z_2 \in \mathbb{C}$, then $|z_1 + z_2| \leq |z_1| + |z_2|$.

Pf

$$\begin{aligned} |z_1 + z_2| &= \sqrt{|z_1 + z_2|^2} \\ &= \sqrt{(z_1 + z_2) \cdot (\overline{z_1 + z_2})} \\ &= \sqrt{(z_1 + z_2) \cdot (\overline{z_1} + \overline{z_2})} \end{aligned}$$

$$= \sqrt{z_1 \cdot \overline{z_1} + z_1 \overline{z_2} + \overline{z_1} z_2 + z_2 \overline{z_2}}$$

$$= \sqrt{|z_1|^2 + z_1 \overline{z_2} + \overline{z_1} z_2 + |z_2|^2}$$

$$= \sqrt{|z_1|^2 + 2 \operatorname{Re}(z_1 \overline{z_2}) + |z_2|^2}$$

$$\leq \sqrt{|z_1|^2 + 2|z_1 \overline{z_2}| + |z_2|^2}$$

$$= \sqrt{|z_1|^2 + 2|z_1||\overline{z_2}| + |z_2|^2}$$

$$= \sqrt{|z_1|^2 + 2|z_1||z_2| + |z_2|^2}$$

$$= \sqrt{(|z_1| + |z_2|)^2}$$

$$= |z_1| + |z_2|$$

□

Lecture 03- Euler's Formula

In the last lecture we saw that every complex number $z = x + iy$ can be written as $z = r(\cos \theta + i \sin \theta)$ where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(\frac{y}{x})$. In this lecture we'll see a very convenient way to represent the $\cos \theta + i \sin \theta$ part of this number, due to Euler.

Let's first recall a few facts from calculus. Under some assumptions about $f(x)$, a function $f(x)$ can be written as

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + f'''(a) \frac{(x-a)^3}{6} + \dots \\ &= \sum_{n=0}^{\infty} f^{(n)}(a) \cdot \frac{(x-a)^n}{n!} \end{aligned}$$

For example, if $f(x) = \cos(x)$ and $a=0$, we have

$$\begin{aligned} \cos(x) &= \cos(0) - \sin(0)x - \cos(0) \frac{x^2}{2} + \sin(0) \frac{x^3}{6} + \cos(0) \frac{x^4}{24} - \sin(0) \frac{x^5}{120} + \dots \\ &= 1 - 0x - \frac{x^2}{2} + 0 \frac{x^3}{6} + \frac{x^4}{24} - 0 \frac{x^5}{120} - \frac{x^6}{720} + \dots \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \end{aligned}$$

Similarly, $\sin(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!}$

$$\text{and } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

In general, the Taylor series for $f(x)$ centered at a ,

$$\sum_{n=0}^{\infty} f^{(n)}(a) \frac{(x-a)^n}{n!}$$

converges in some interval around a (the series' interval of convergence), though for e^x , $\cos(x)$, and $\sin(x)$ these series will converge for all x in $(-\infty, \infty)$.

Recall also that a series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent if $\sum_{n=0}^{\infty} |a_n|$ converges. Absolute convergence is stronger than just convergence since it implies convergence, but convergence does not imply absolute convergence. If a series converges but is not absolutely convergent, the series is called conditionally convergent.

Here are two important theorems about absolutely and conditionally convergent series you may not be aware of, but which are extremely important.

Thm

If $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, and if a'_n is some rearrangement of the a_n terms, then $\sum_{n=0}^{\infty} a'_n$ is also absolutely convergent and $\sum a_n = \sum a'_n$. $\square$

Thm

If $\sum b_n$ is conditionally convergent and $\sum b_n = L$, then for every $M \in \mathbb{R}$ there exists some rearrangement b'_n of the b_n terms so that $\sum b'_n = M$. □

That is, if a series is absolutely convergent then its terms can be rearranged and the value of the series is not changed, but this is very false if the series is only conditionally convergent.

The proof of these theorems are a bit technical (they're really part of real analysis), so we'll post them at the end of these notes and leave them as optional material you can think through if you're interested.

Let's now try to use the Taylor series above to see if we can determine what $e^{i\theta}$ should be. First, we will define $e^{i\theta}$ using the same series as e^x for real x . Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we'll define $e^{i\theta}$ as

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n \theta^n}{n!}$$

Now, using that $|z_1 z_2| = |z_1| |z_2|$ — a fact whose proof we'll leave as an (easy) exercise, let's observe

$$\sum_{n=0}^{\infty} \left| \frac{i^n \theta^n}{n!} \right| = \sum_{n=0}^{\infty} |i|^n \left| \frac{\theta^n}{n!} \right| = \sum_{n=0}^{\infty} \left| \frac{\theta^n}{n!} \right|$$

This series must converge as the power series of e^θ is absolutely convergent. Hence $e^{i\theta}$ is also absolutely convergent. Thus we may rearrange the terms of the series representing $e^{i\theta}$.

Before rearranging the terms, let us observe

$$\begin{aligned} i^0 &= 1, & i^1 &= i, & i^2 &= -1, & i^3 &= -i, \\ i^4 &= 1, & i^5 &= i, & i^6 &= -1, & i^7 &= -i, \\ && \text{and so on.} \end{aligned}$$

We'll use this to rearrange the terms of $e^{i\theta} = \sum_{n=0}^{\infty} i^n \frac{\theta^n}{n!}$ by grouping the real and purely imaginary terms together

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} i^n \frac{\theta^n}{n!} = 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \dots \\ &= 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots + i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - i\frac{\theta^7}{7!} + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} \end{aligned}$$

But notice these last two series are precisely the Taylor series representation of $\cos \theta$ and $\sin \theta$ centered at 0, and so we now have the famous formula of Euler,

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Notice in the case $\theta = \pi$ this becomes

$$e^{i\pi} = \cos \pi + i \sin \pi = -1$$

$$\text{or, } e^{i\pi} + 1 = 0.$$

Since we had previously observed every complex number can be written as $r(\cos \theta + i \sin \theta)$, we are now justified in claiming each $z \in \mathbb{C}$ may be written as $z = r e^{i\theta}$.

Observe multiplication of complex numbers $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ is particularly nice in this setting,

$$z_1 z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

agreeing with our earlier statement that arguments of complex numbers are added when the numbers are

multiplied.

Applying this to raising a number to an (integer) power,

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

⋈

The following theorem is optional, but it justifies our claim that we can rearrange terms in an absolutely convergent series.

Then

If $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series converging to L , and if $\sum_{n=1}^{\infty} a'_n$ is any rearrangement of the a_n terms (i.e., each $a'_n = a_{m_n}$ for some m —more precisely, there is some bijection $\sigma: \mathbb{N} \rightarrow \mathbb{N}$ so that $a'_n = a_{\sigma(n)}$), then $\sum_{n=1}^{\infty} a'_n$ is also absolutely convergent and also converges to L .

Pf

By assumption, $\sum_{n=1}^{\infty} |a_n|$ converges, which means the sequence of partial sums $S_N = \sum_{n=1}^N |a_n|$ is a convergent, monotonic increasing sequence with $S_N \rightarrow M$ for some $M \in \mathbb{R}$. Note $S_N \leq M$ for every N .

Now consider the series $\sum_{n=1}^{\infty} |a'_n|$ and correspondingly partial sums $S'_N = \sum_{n=1}^N |a'_n|$. Note S'_N is also a monotonic increasing sequence. Observe each $S'_N \leq M$ as each $S'_N \leq S_K$ for some K , and $S_K \leq M$. As S'_N is increasing and bounded, it must converge. Thus $\sum_{n=1}^{\infty} a'_n$ is also absolutely convergent.

We still need to show $\sum_{n=1}^{\infty} a'_n = L$. Equivalently, we may show for every $\epsilon > 0$ there exists an M so that if $N > M$, $|\sum_{n=1}^N a'_n - L| < \epsilon$.

Note there is an M_1 such that if $N \geq M_1$, then

$$\left| \sum_{n=1}^N a_n - L \right| < \epsilon/2 \quad \text{and} \quad \sum_{n=N+1}^{\infty} |a_n| < \epsilon/2.$$

Now, we may choose $M \geq M_1$ so that if $N \geq M$, then $\{a_1, \dots, a_{M_1}\} \subseteq \{a'_1, \dots, a'_N\}$. Define D_N as the set $\{a'_1, \dots, a'_N\} \setminus \{a_1, \dots, a_{M_1}\}$. We then have

$$\left| \sum_{n=1}^{\infty} a'_n - L \right| = \left| \sum_{n=1}^{M_1} a_n + \sum_{n \in D_N} a'_n - L \right|$$

$$\leq \left| \sum_{n=1}^{M_1} a_n - L \right| + \left| \sum_{n \in D_N} a'_n \right|$$

$$\leq \left| \sum_{n=1}^{M_1} a_n - L \right| + \sum_{n \in D_N} |a'_n|$$

$$\leq \left| \sum_{n=1}^{M_1} a_n - L \right| + \sum_{n=M_1+1}^{\infty} |a_n|$$

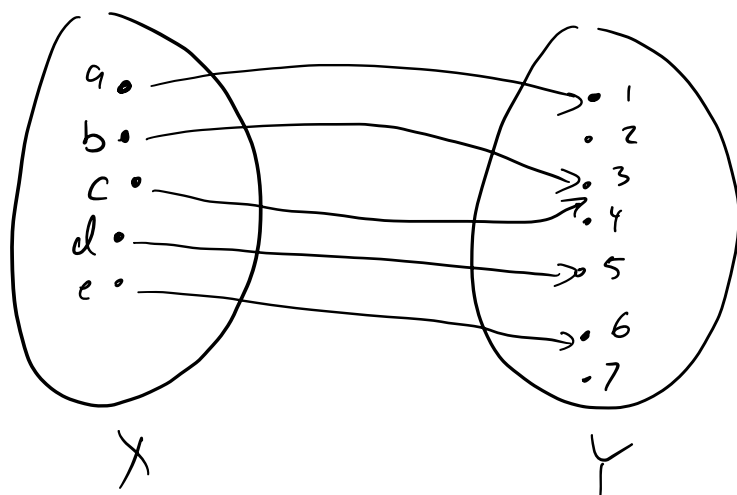
$$< \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon.$$

□

Math 424, Spring 2024
Lecture 04 - Complex Mappings

Recall that a map between sets X and Y is an association of elements in Y to those in X such that each element of X has one element of Y associated to it. Formally this is a subset of pairs of $X \times Y$, say $f \subseteq X \times Y$ such that for every $x \in X$ there is exactly one $(x, y) \in f$. We usually think of f as mapping an "input" $x \in X$ to an "output" $y \in Y$ denoted $f(x)$. We call X the domain of f , Y is the codomain of f , and the set of all $y \in Y$ such that $y = f(x)$ for some $x \in X$ is called the image of f , denoted $f(X)$.



$$f(X) = \{1, 3, 4, 5, 6\}$$

By a complex mapping we simply mean a function of the form $f: D \rightarrow \mathbb{C}$ where D is some subset of $\mathbb{C}$.

For example, $f: \mathbb{C} \rightarrow \mathbb{C}$ given by $f(z) = z^3$ is one such map; as is $g: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ given by

$$g(z) = \frac{1}{z}.$$

Since each complex number may be expressed as $z = x + iy$, the outputs of a function $f: D \rightarrow \mathbb{C}$ has real and imaginary parts which themselves are functions of x and y , thus we sometimes will find it convenient to think of f as a function

$$f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)$$

In this way we can think of a map $f: D \rightarrow \mathbb{C}$ as a pair of maps $u, v: D \rightarrow \mathbb{R}$.

For example, the map $f(z) = z^3$ can be expressed as follows: since

$$\begin{aligned} z^3 &= (x + iy)^3 = x^3 + 3x^2iy + 3x(iy)^2 + (iy)^3 \\ &= x^3 + i3x^2y - 3xy^2 - iy^3 \\ &= (x^3 - 3xy^2) + i(3x^2y - y^3) \end{aligned}$$

Here we have $f(z) = z^3$ corresponds to $f(z) = u(x, y) + iv(x, y)$ where

$$\begin{aligned} u(x, y) &= x^3 - 3xy^2 \\ v(x, y) &= 3x^2y - y^3 \end{aligned}$$

For $f(z) = \frac{1}{z}$, we have

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right)$$

and so $f(z) = \frac{1}{z}$ may be written as
 $f(z) = u(x,y) + iv(x,y)$ where

$$u(x,y) = \frac{x}{x^2+y^2} \quad v(x,y) = \frac{-y}{x^2+y^2}$$

We can also take a pair of functions
 $u, v: D \rightarrow \mathbb{R}$ and, interpreting them as determining
a function $u+iv: D \rightarrow \mathbb{C}$, rewrite that function
in terms of z and $\bar{z}$.

For example, if u and v are given by

$$u(x,y) = y^2 - x^2 + 1$$

$$v(x,y) = -2xy$$

how can we express $u+iv$ as a function of z ?

For an input $z = x+iy$, note $x = \frac{z+\bar{z}}{2}$ and
 $y = \frac{z-\bar{z}}{2i}$, so let's rewrite $u(x,y) + iv(x,y)$ by
replacing x and y by these expressions:

$$\begin{aligned} u(x,y) + iv(x,y) &= y^2 - x^2 + 1 - i2xy \\ &= \left(\frac{z-\bar{z}}{2i}\right)^2 - \left(\frac{z+\bar{z}}{2}\right)^2 + 1 - i2\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right) \end{aligned}$$

$$= \frac{z^2 - 2z\bar{z} + \bar{z}^2}{-4} - \frac{z^2 + 2z\bar{z} + \bar{z}^2}{4} + 1 - \frac{z^2 - \bar{z}^2}{2}$$

$$= \frac{1}{4} \left(-z^2 + 2z\bar{z} - \bar{z}^2 - z^2 - 2z\bar{z} - \bar{z}^2 + 4 - 2z^2 + 2\bar{z}^2 \right)$$

$$= \frac{1}{4} \left(-4z^2 + 0z\bar{z} + 0\bar{z}^2 + 4 \right)$$

$$= -z^2 + 1$$

As another example, consider the map $u+iv$ given by

$$u(x,y) = 3x^2 + x - 3y^2$$

$$v(x,y) = y - 6xy$$

Thus

$$f(z) = u(x,y) + iv(x,y)$$

$$= 3x^2 + x - 3y^2 + i(y - 6xy)$$

$$= 3 \left(\frac{z+\bar{z}}{2} \right)^2 + \frac{z+\bar{z}}{2} - 3 \left(\frac{z-\bar{z}}{2i} \right)^2 + i \left(\frac{z-\bar{z}}{2i} - 6 \frac{z+\bar{z}}{2} \cdot \frac{z-\bar{z}}{2i} \right)$$

$$= 3 \frac{z^2 + 2z\bar{z} + \bar{z}^2}{4} + \frac{z+\bar{z}}{2} - 3 \frac{z^2 - 2z\bar{z} + \bar{z}^2}{-4} + \frac{z-\bar{z}}{2} - 3 \left(\frac{z^2 - \bar{z}^2}{2} \right)$$

$$= \frac{1}{4} \left(3z^2 + 6z\bar{z} + 3\bar{z}^2 + 2z + 2\bar{z} + 3z^2 - 6z\bar{z} + 3\bar{z}^2 + 2z - 2\bar{z} - 6z^2 + 6\bar{z}^2 \right)$$

$$= \frac{1}{4} \left(12\bar{z}^2 + 4z \right) = 3\bar{z}^2 + z$$

It will also sometimes be convenient to express a complex mapping in polar coordinates. That is, writing $f(z) = u(z) + iv(z)$, sometimes we may find it useful to describe u and v using z 's polar form, $z = re^{i\theta}$.

For instance the map $z \mapsto iz^3$ may be expressed as

$$\begin{aligned} iz^3 &= i(re^{i\theta})^3 \\ &= i \cdot r^3 e^{i3\theta} \\ &= r^3 e^{i(3\theta + \pi/2)} \\ &= \underbrace{r^3 \cos(3\theta + \pi/2)}_{u(r, \theta)} + i \underbrace{r^3 \sin(3\theta + \pi/2)}_{v(r, \theta)} \end{aligned}$$

In short, there are several ways we may choose to specify a complex map, and each option will be useful or convenient for certain types of maps.

Recall from basic algebra and calculus that associated to a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is a graph, a subset of $\mathbb{R}^2$ given as $\{(x, y) \in \mathbb{R}^2 \mid y = f(x)\}$. As you know from previous courses, graphs are often helpful for visualizing a function and can be a useful tool for discerning various properties of the function.

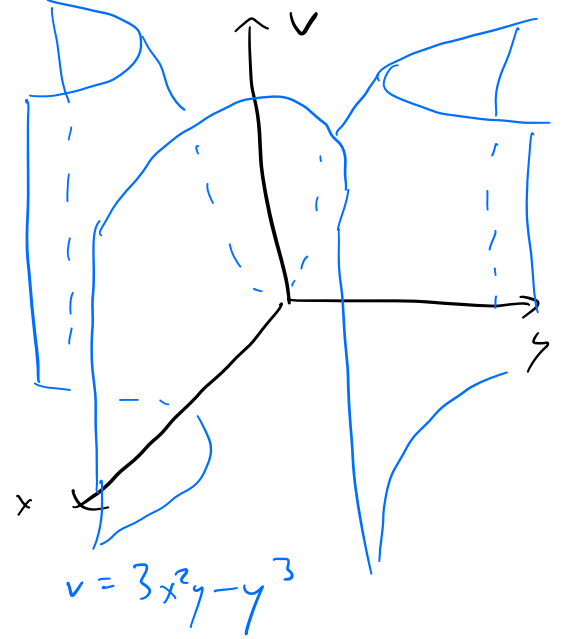
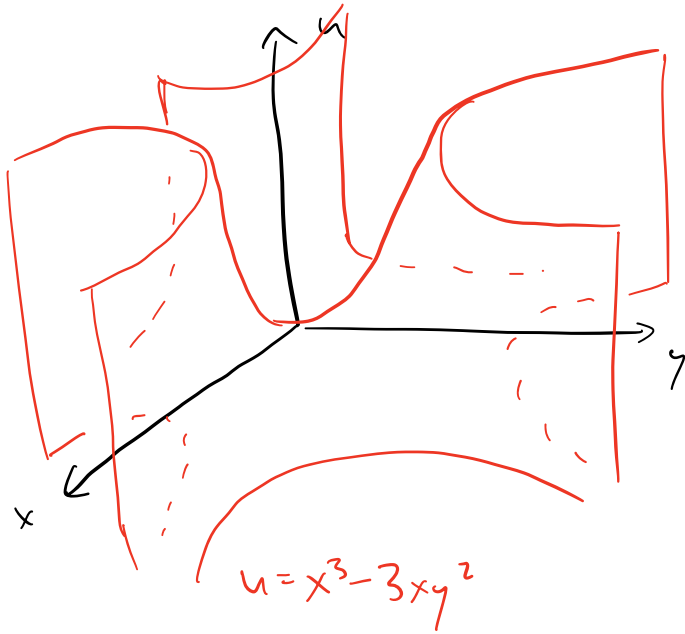
We can similarly define the graph of a complex function $f: \mathbb{C} \rightarrow \mathbb{C}$ as a subset of $\mathbb{C}^2$,

$$\{(z, w) \in \mathbb{C}^2 \mid w = f(z)\}$$

but we can't easily visualize this set, as $\mathbb{C}^2$ has four (real) dimensions. Still, it would be nice if we had some way of graphically describing a complex map, and there are a few ways of doing this.

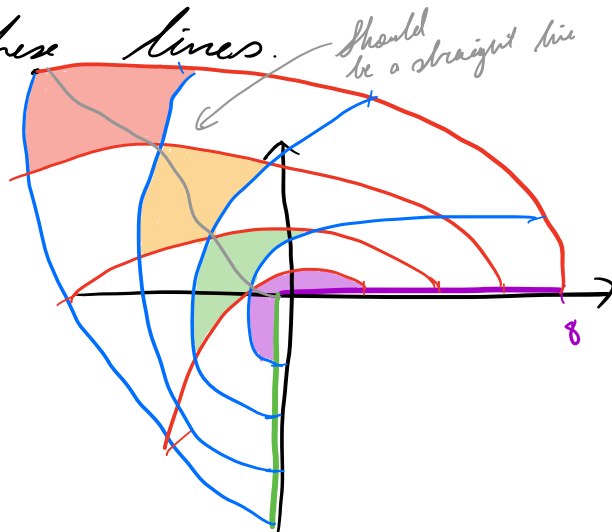
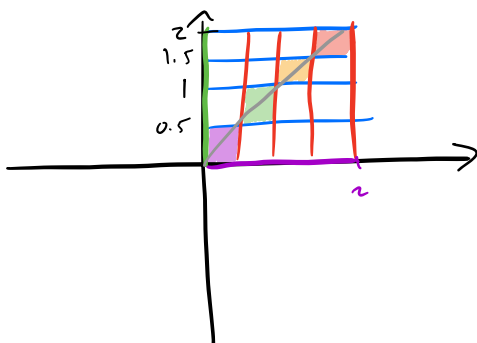
One way would be to graph the real and imaginary parts separately. That is, if we write $f(z) = u(z) + iv(z)$, and identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way, mapping $x + iy$ to (x, y) , then we can think of u and v as two real-valued functions on a subset of the plane, and consider their graphs as subsets of $\mathbb{R}^3$.

For example, if $f(z) = z^3$, then we had determined earlier that this is $f = u + iv$ where $u(x, y) = x^3 - 3xy^2$ and $v(x, y) = 3x^2y - y^3$. These functions each determine surfaces in $\mathbb{R}^3$.



Though this is a reasonable way to graph $f(z) = z^3$, there is another way to visualize the function's action on $\mathbb{C}$ by interpreting it as a transformation of $\mathbb{C}$.

That is, we will draw some subset of $\mathbb{C}$ in one copy of the plane, and in a second copy draw the image of that subset. For instance, our set in $\mathbb{C}$ might be a collection of horizontal and vertical lines, and we will draw the images of these lines.



Notice each line on the left can be parametrized, for example the "middle" horizontal line is $t+i$ for $0 \leq t \leq 2$ (or $(t,1)$ in Cartesian coordinates on $\mathbb{R}^2$). We are mapping points $x+iy$ (or (x,y)) to $x^3-3xy^2+i(3x^2y-y^3)$ (or, $(x^3-3xy^2, 3x^2y-y^3)$), so our line $t+i$ ($(t,1)$), gets sent to $t^3-3t+i(3t^2-1)$ — or, $(t^3-3t, 3t^2-1)$ for $0 \leq t \leq 2$.

As another example, consider the mapping $f(z) = 1-z^2$. In Cartesian coordinates, this is the map

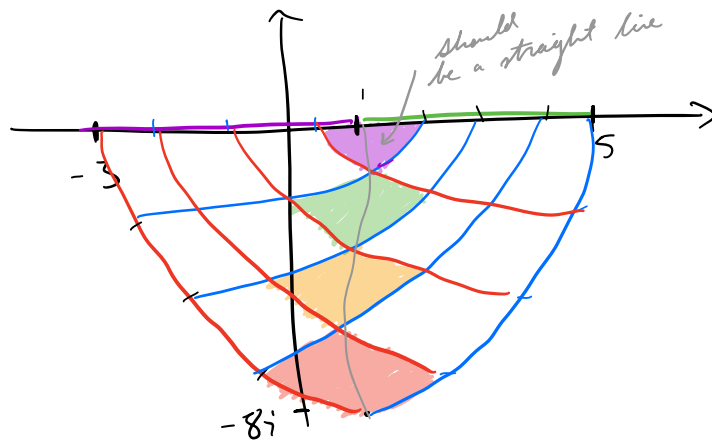
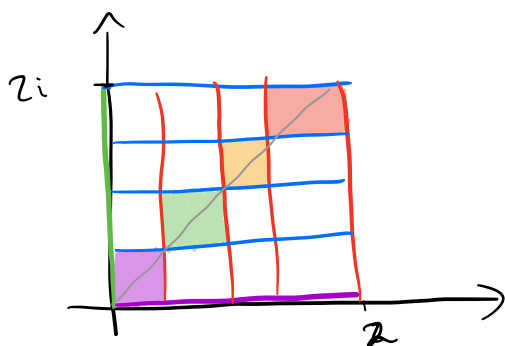
$$(x,y) \mapsto (-x^2+y^2+1, -2xy)$$

as

$$\begin{aligned} 1-z^2 &= 1-(x+iy)^2 \\ &= 1-(x^2+i2xy-y^2) \\ &= (-x^2+y^2+1) + i(-2xy) \end{aligned}$$

This maps a horizontal line (t,k) , where k is some fixed number, so this is the line $y=k$, to the curve $(-t^2+k^2+1, -2tk)$; a vertical line (l,t) gets sent to $(-l^2+t^2+1, -2lt)$.

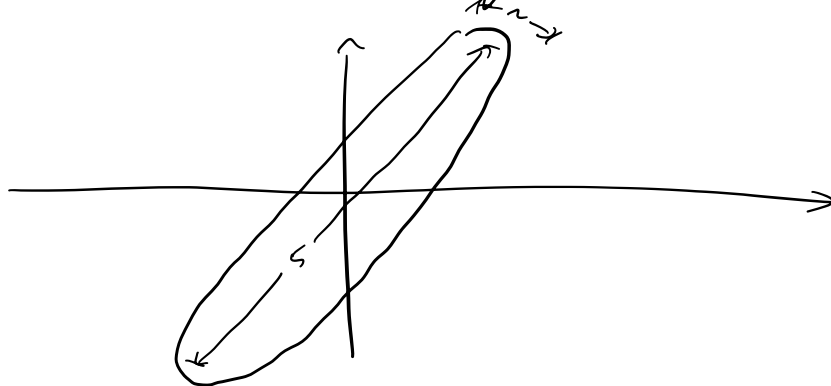
Mapping a few horizontal & vertical lines to their images, we get the following kind of picture:



We can use these ideas to help us find parametrizations of curves in the plane:

Example

Determine the parametrization of a rotated ellipse, rotated 60° with major axis of length 5 and minor axis of length 2.



We can easily parametrize the non-rotated ellipse as $(5\cos(t), 2\sin(t))$, $0 \leq t \leq 2\pi$.

Rotating by 60° means multiplying by $e^{i\pi/3}$, so we're considering the image of the complex map $F(z) = e^{i\pi/3} z$. The Cartesian coordinates this map is

$$x+iy \mapsto (\cos(\pi/3) + i\sin(\pi/3))(x+iy)$$

$$\begin{aligned}
&= \cos(\pi/3)x + i\cos(\pi/3)y + i\sin(\pi/3)x - \sin(\pi/3)y \\
&= \left[\cos(\pi/3)x - \sin(\pi/3)y \right] + i \left[\sin(\pi/3)x + \cos(\pi/3)y \right]
\end{aligned}$$

So, in the plane, rotation by $\pi/3$ is accomplished by

$$(x, y) \mapsto (\cos(\pi/3)x - \sin(\pi/3)y, \sin(\pi/3)x + \cos(\pi/3)y)$$

Rmk

Writing (x, y) as a column vector $\begin{pmatrix} x \\ y \end{pmatrix}$, this rotation is expressed as multiplying by the matrix $\begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{pmatrix}$ — of course, we can replace $\pi/3$ by any angle θ .

Now applying this map to the ellipse parametrized by $(5\cos(t), 2\sin(t))$, we have the rotated ellipse is parametrized by

$$(5\cos(\pi/3)\cos(t) - 2\sin(\pi/3)\sin(t), 5\sin(\pi/3)\cos(t) + 2\cos(\pi/3)\sin(t))$$

for $t \in [0, 2\pi]$.

Let's observe that the 3 types of isometries of the Euclidean plane have nice interpretations as complex mappings:

- Rotation counter-clockwise by θ corresponds to the map $z \mapsto e^{i\theta} z$
- Translation by a vector $\begin{pmatrix} a \\ b \end{pmatrix}$ corresponds to $z \mapsto z + a + ib$
- Reflection across the x -axis corresponds to $z \mapsto \bar{z}$

HW

What about reflecting across a line $ax + by = c$?

Let's also summarize what the map $z \mapsto z^n$ represents. Using Euler's formula to write $z = r e^{i\theta}$, the map sends this to $z^n = r^n e^{in\theta}$, so the absolute value is raised to n , and the angle is multiplied by n . Let's use this to determine the solutions to the equation $z^n = 1$ (i.e., to determine the n^{th} -roots of 1).

If $z = r e^{i\theta}$ solves $z^n = 1$, then $r^n e^{in\theta} = 1$, or $r^n e^{in\theta} = 1 \cdot e^{i0}$, so the absolute value to the n^{th} power is 1. Assuming $r > 0$, the only solution to $r^n = 1$ is $r = 1$. So, our solutions to $z^n = 1$ have absolute value 1 and must lie on the unit circle S^1 . The angle θ must be some value so that $n\theta = 0$ — or an equivalent angle such as $2\pi, 4\pi, 6\pi \dots$

That is $n\theta = 2\pi m$ for some $m \in \mathbb{Z}$, so
 $\theta = \frac{2\pi}{n}m$ for some integer m . Then θ
 may be

$$\dots, \frac{-6\pi}{n}, \frac{-4\pi}{n}, \frac{-2\pi}{n}, 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \frac{6\pi}{n}, \dots$$

Now, how many of these are actually distinct angles? First observe the values

$$0, \frac{2\pi}{n}, \frac{2\pi}{n} \cdot 2, \frac{2\pi}{n} \cdot 3, \dots, \frac{2\pi}{n} \cdot (n-1)$$

are all distinct angles for $n > 1$ as each one is a different number in $[0, 2\pi)$. The multiples after this, though, give us one of these angles:

$$2\pi = \frac{2\pi}{n} \cdot n \cong 0$$

$$\frac{2\pi}{n} \cdot (n+1) = 2\pi + \frac{2\pi}{n} \cong \frac{2\pi}{n}$$

$$\frac{2\pi}{n} \cdot (n+2) = 2\pi + \frac{2\pi}{n} \cdot 2 \cong \frac{2\pi}{n} \cdot 2$$

etc.

Similarly for the negative multiples:

$$\frac{2\pi}{n} \cdot (-1) \cong 2\pi - \frac{2\pi}{n} = (n-1) \cdot \frac{2\pi}{n}$$

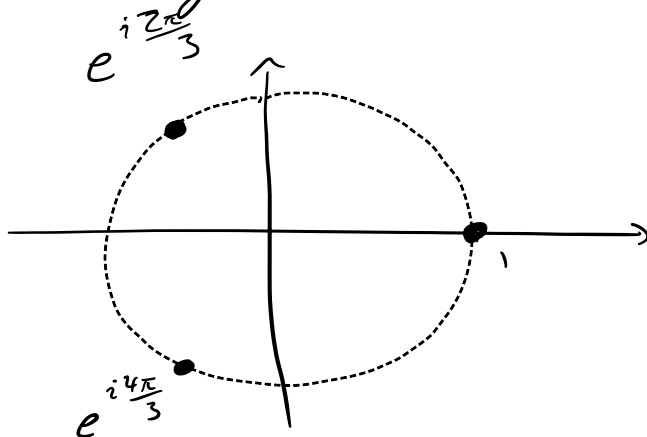
$$\frac{2\pi}{n} \cdot (-2) \cong 2\pi - \frac{2\pi}{n} \cdot 2 = (n-2) \cdot \frac{2\pi}{n}$$

etc.

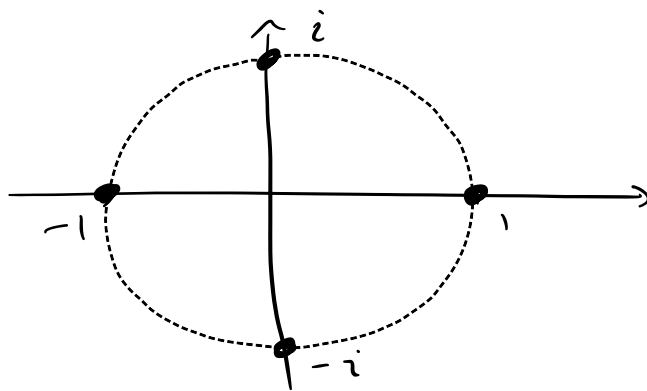
Thus there are exactly n solutions to $z^n = 1$, given by $e^{i \frac{2\pi k}{n}}$, $k=0, 1, \dots, n-1$.

$z^2 = 1$, for example, is solved by $e^{i \frac{2\pi \cdot 0}{2}}$, $e^{i \frac{2\pi \cdot 1}{2}}$ which are just 1 and -1.

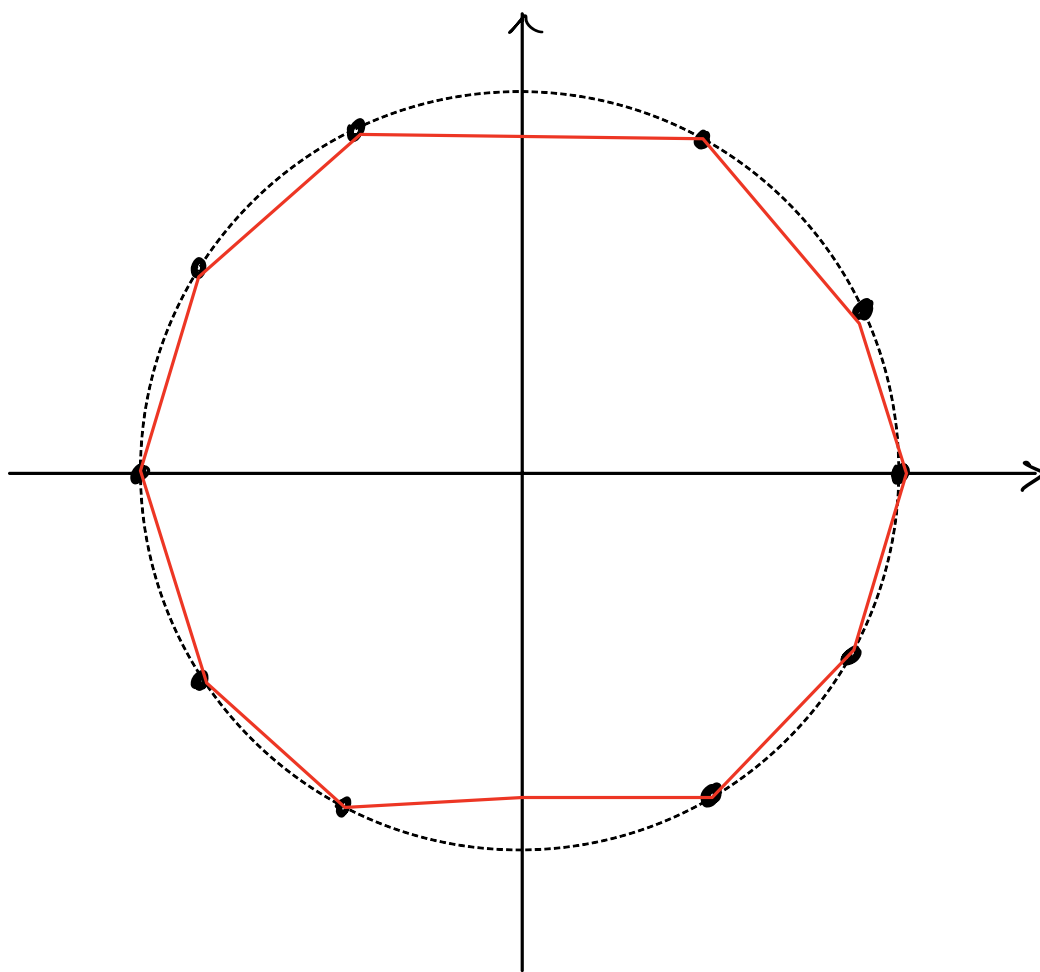
$z^3 = 1$ is solved by 1, $e^{i \frac{2\pi}{3}}$, $e^{i \frac{4\pi}{3}}$



$z^4 = 1$ is solved by 1, $e^{i \frac{2\pi}{4}}$, $e^{i \frac{4\pi}{4}}$, $e^{i \frac{6\pi}{4}}$ which are 1, i , -1 , $-i$.



The n solutions of $z^n = 1$ are called the n^{th} roots of unity and are the vertices of the regular n -gon inscribed in the unit circle with a vertex at 1.



The tenth roots of unity, for example, occur at the vertices of a regular decagon.

Exercise for students w/ some abstract algebra

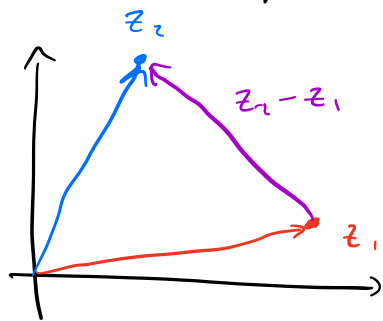
Show the n^{th} roots of unity form a group under complex multiplication

Math 424, Spring 2024
Lecture 05 - Some Topology

Before discussing complex maps some more, it would be beneficial for us to have some basic topological ideas at our disposal.

Let's begin by noticing there is a natural way to measure distance between two complex numbers:

given $z_1, z_2 \in \mathbb{C}$, we think of the vectors from 0 to these points. Notice $z_2 - z_1$ is the vector between their tips - i.e., from z_1 to z_2 :



The distance between z_1 and z_2 is then the length of this vector, $|z_2 - z_1|$.

In terms of the real and imaginary parts of the complex numbers, say $z_j = x_j + iy_j$ for $j=1,2$, we can compute this as

$$\begin{aligned} |z_2 - z_1| &= |(x_2 - x_1) + i(y_2 - y_1)| \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \end{aligned}$$

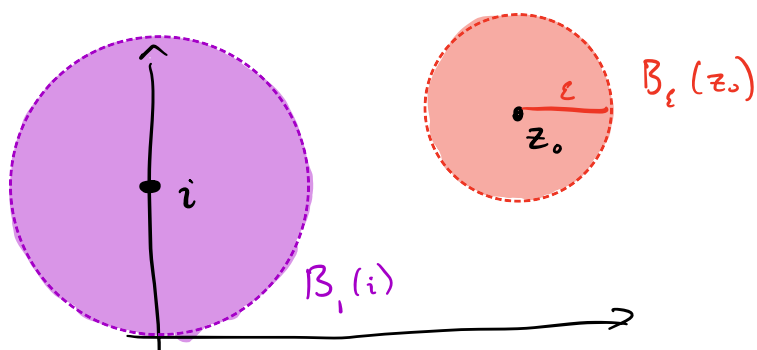
E.g., the distance between $2+7i$ and $5+3i$ is

$$|(5+3i) - (2+7i)| = |3-4i| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25}$$

$$= 5$$

For a real number $\varepsilon > 0$, we define the open ε -ball around $z_0 \in \mathbb{C}$ to be the following subset of $\mathbb{C}$, denoted $B_\varepsilon(z_0)$:

$$B_\varepsilon(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\}$$



A subset $U \subseteq \mathbb{C}$ is called open if it can be expressed as a union of open balls. i.e., $U \subseteq \mathbb{C}$ is open if for each $u \in U$ there exists some $\varepsilon_u > 0$ so that

$$U = \bigcup_{u \in U} B_{\varepsilon_u}(u)$$

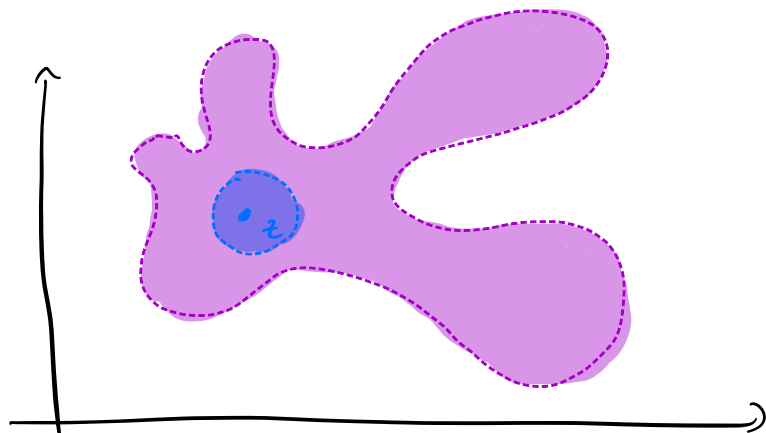
Thm

$U \subseteq \mathbb{C}$ is open if and only if for each

$u \in U$ there exists an $\varepsilon > 0$ so that $B_\varepsilon(u) \subseteq U$.
Pf
 Homework

Thm
 Every union of open subsets
 of $\mathbb{C}$ is open.

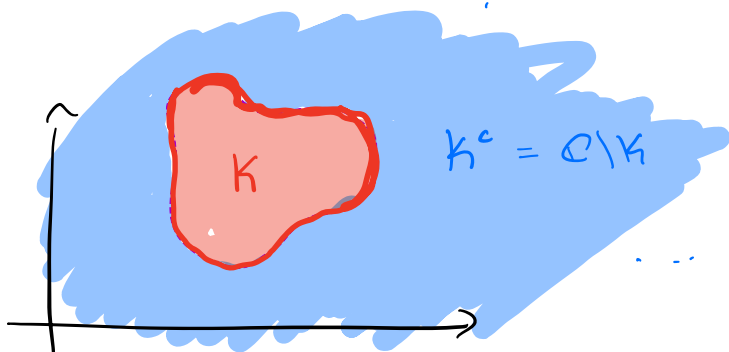
Pf
 Let $U_i, i \in I$, be open. $\bigcup_{i \in I} U_i = \bigcup_{i \in I} \left(\bigcup_{u \in U_i} B_{\varepsilon_u}(u) \right)$. $\square$



A subset $K \subseteq \mathbb{C}$ is closed if its complement is open.

Thm
 Every intersection of closed
 subsets of $\mathbb{C}$ is closed.

Pf
 $\bigcap_{i \in I} K_i = \left(\bigcup_{i \in I} K_i^c \right)^c$. $\square$



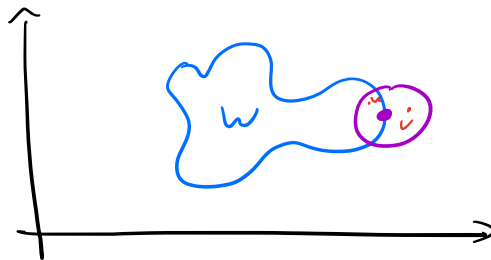
Question for students

Which of the following sets are open? Which are closed?

- $\{z \in \mathbb{C} \mid \operatorname{Re}(z) + \operatorname{Im}(z) \geq 1\}$ — closed, not open
- $\{z \in \mathbb{C} \mid |z| > 2\}$ — open, not closed
- $\{z \in \mathbb{C} \mid |z| \geq 2\}$ — closed, not open
- $\{z \in \mathbb{C} \mid 0 \leq \operatorname{Re}(z) \leq 1, \operatorname{Im}(z) = 0\}$ — closed, not open
- $\{z \in \mathbb{C} \mid 0 < \operatorname{Re}(z) < 1, \operatorname{Im}(z) = 0\}$ — neither open nor closed
- $\{0, 1, i\}$ — closed, not open

- $\mathbb{C}$ — both open & closed
- $\emptyset$ — both open & closed

A point $z \in \mathbb{C}$ is called a boundary point of a subset $W \subseteq \mathbb{C}$ if $\forall \varepsilon > 0$, there exists points $u, v \in B_\varepsilon(z)$ such that $u \in W$ and $v \notin W$.



The collection of all boundary points of W is called the boundary of W , denoted ∂W .

Thm
 $W \subseteq \mathbb{C}$ is closed if and only if $\partial W \subseteq W$.

Pf
 Homework.

Q

What's the boundary of each of sets in the example of open & closed sets above?

The closure of a set $W \subseteq \mathbb{C}$ denoted $\text{cl}(W)$ (sometimes this is also denoted $\bar{W}$, but we'll avoid this notation to prevent any confusion w/conjugation) is the union of W and its boundary, $\text{cl}(W) = W \cup \partial W$.

An easy corollary of the theorem above is that a set W is closed iff $W = cl(W)$.

The interior of $W \subseteq \mathbb{C}$ is the set of $w \in W$ such that there is an open set $U \subseteq \mathbb{C}$ so that $w \in U \subseteq W$. The interior of W is denoted $int(W)$ or W° .

Notice $int(W)$ is an open set: for each $w \in int(W)$, let U_w denote an open set such that $w \in U_w \subseteq W$. Then $int(W) = \bigcup_{w \in int(W)} U_w$.

Thm

$int(W)$ is the largest open set contained in W .

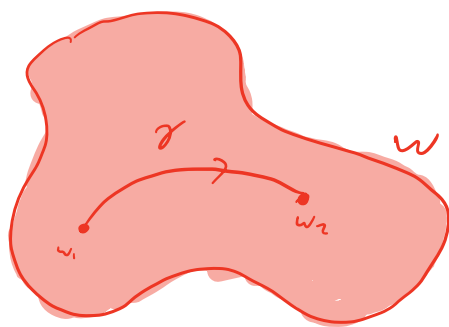
Pf

Suppose V was some larger open set contained in W for the sake of contradiction. i.e., suppose $int(W) \subsetneq V \subseteq W$. Let $v \in V$. Since V is open, there is some $\varepsilon > 0$ so that $B_\varepsilon(v) \subseteq V \subseteq W$, but then, as $B_\varepsilon(v)$ is an open subset of W , we must have $v \in int(W)$, so $V \subseteq int(W)$, but this is a contradiction to our assumption $int(W) \subsetneq V$. $\square$

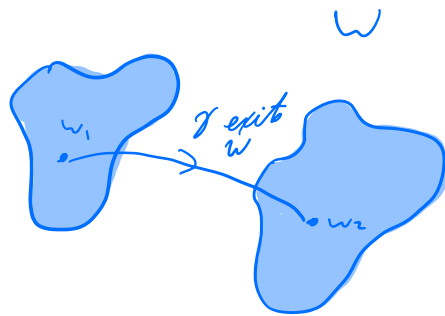
Math 424, Spring 2024
Lecture 06 - Some More Topology

In this lecture we want to use some of the ideas from the previous lecture to define and study special maps called "continuous" maps. Before jumping into continuous maps, we'll need a couple more definitions that will be helpful later.

We'll say a set $W \subseteq \mathbb{C}$ is connected if for every pair of points $w_1, w_2 \in W$, there is a curve γ which starts at w_1 , ends at w_2 , and always remains inside W .



Connected



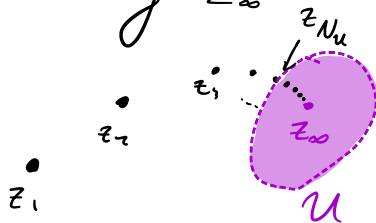
Not connected

While the idea of connectedness seems simplistic, it will turn out to be a very useful property—certain "nice" things will happen if a set we care about (e.g., the domain of some complex map we want to study) is connected.

It will sometimes be helpful to describe various

properties in terms of sequences of complex numbers. Here, a sequence is just a map $\mathbb{N} \rightarrow \mathbb{C}$ where we often write z_n for the image of $n \in \mathbb{N}$ under such a map.

Given a sequence of complex numbers, $(z_n)_{n \in \mathbb{N}}$, we say $z_\infty \in \mathbb{C}$ is the limit of the sequence if for every open set U containing z_∞ there exists some $N_U \in \mathbb{N}$ such that for all $n > N_U$, $z_n \in U$.



The limit of the sequence, provided it exists, is denoted $L = \lim_{n \rightarrow \infty} z_n$.

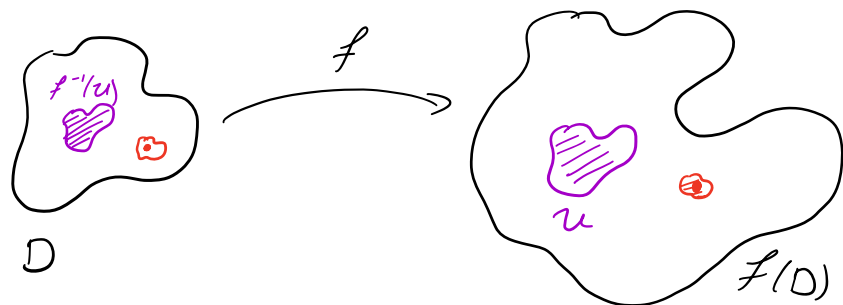
Thm

Suppose $(z_n)_{n \in \mathbb{N}}$ is a sequence of complex numbers. Then $\lim_{n \rightarrow \infty} z_n = z_\infty$ if and only if for every $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ such that for all $n > N_\varepsilon$, $|z_\infty - z_n| < \varepsilon$.

Pf
 Suppose for all $\varepsilon > 0$ there is an N_ε such that $n > N_\varepsilon \Rightarrow |z_\infty - z_n| < \varepsilon$. For any open set U containing z_∞ there exists an ε -ball around z_∞ within U , so $z_n \in U$ for all $n > N_\varepsilon$.

Suppose for every open set U around z_∞ , $\exists N_U$ so that $n > N_U \Rightarrow z_n \in U$. Consider $U = B_\varepsilon(z_\infty)$. $\square$

A complex map $f: D \rightarrow \mathbb{C}$ is continuous if for every open $U \subseteq \mathbb{C}$, the preimage $f^{-1}(U) \subseteq D$ is open.



Thm

$f: D \rightarrow \mathbb{C}$ is continuous if for every point $z \in D$ and every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $w \in D$ satisfying $|z - w| < \delta$, $|f(z) - f(w)| < \varepsilon$.

Pf

Homework. (Hint: This condition, $|z - w| < \delta \Rightarrow |f(z) - f(w)| < \varepsilon$ can be rephrased as $w \in B_\delta(w) \Rightarrow f(w) \in B_\varepsilon(f(z))$. Try to rephrase this in terms of the preimage $f^{-1}(B_\varepsilon(f(z)))$. □

We can also rephrase continuity in terms of sequences.

Thm

A map $f: D \rightarrow \mathbb{C}$ is continuous if and only if for every sequence $(z_n)_{n \in \mathbb{N}}$ contained in D converging to $z_\infty \in D$, we have that the sequence $(f(z_n))_{n \in \mathbb{N}}$ converges to $f(z_\infty)$.

If
 Suppose f is continuous and $(z_n)_{n \in \mathbb{N}}$ and z_∞ are as described in the theorem. Let U be any open subset containing $f(z_\infty)$. Then $f^{-1}(U)$ is an open set containing z_∞ , so there is some $N \in \mathbb{N}$ such that for all $n > N$, $z_n \in f^{-1}(U)$. Hence $f(z_n) \in U$, so $f(z_n) \rightarrow f(z_\infty)$.

Now suppose for every sequence $z_n \rightarrow z_\infty$ in D we have $f(z_n) \rightarrow f(z_\infty)$. Let U be any open subset of $\mathbb{C}$. We must show $f^{-1}(U) \subseteq D$ is open. If $f^{-1}(U) = \emptyset$, then we're done. So suppose $f^{-1}(U) \neq \emptyset$. Let z_∞ be any point in $f^{-1}(U)$.

We claim there is an $\varepsilon > 0$ so that $B_\varepsilon(z_\infty) \subseteq f^{-1}(U)$. If not, there exists a sequence z_n in D with $z_n \rightarrow z_\infty$, yet $z_n \notin f^{-1}(U)$. But this would mean $f(z_n) \notin U$, hence $f(z_n)$ can not converge to $f(z_\infty) \in U$, and we have a contradiction.

Thus every point in $f^{-1}(U)$ has an open ball around it contained in $f^{-1}(U)$, and so $f^{-1}(U)$ is open. □

The way you should think about continuous maps is that they take "nearby" points to "nearby"

points — if w, z are "close," then $f(w), f(z)$ are "close."

It will turn out that continuous maps are the maps we care about in topology. (The fancy way to say this is that continuous maps are the morphisms in the category of topological spaces.) In complex analysis the maps we'll really care about will be continuous, together with some stronger properties, so it will be beneficial to us to have some understanding of continuous maps in general.

First let's verify a few particularly simple complex maps are continuous.

Thm

The constant map $f: \mathbb{C} \rightarrow \mathbb{C}$ sending all $z \in \mathbb{C}$ to some fixed constant $k \in \mathbb{C}$ is continuous.

Pf

Let $U \subseteq \mathbb{C}$ be an open set. We must verify that the preimage $f^{-1}(U)$ is also open. We will do this by considering two cases: if $k \in U$ or $k \notin U$.

If $k \notin U$, then $f^{-1}(U) = \emptyset$ which is open.

If $k \in U$, then $f^{-1}(U) = \mathbb{C}$ which is open.

□

Thm

The identity map $f: \mathbb{C} \rightarrow \mathbb{C}$ given by $f(z) = z$ is continuous

Pf

Let $U \subseteq \mathbb{C}$ be open. Note $f^{-1}(U) = U$, so $f^{-1}(U)$ is open. □

Now let's observe that products and sums of continuous maps are also continuous

Thm

If $f, g: D \rightarrow \mathbb{C}$ are continuous, then so are $f \cdot g$ (ie, the map $z \mapsto f(z) \cdot g(z)$) and $f + g$ (the map $z \mapsto f(z) + g(z)$).

Pf

Let z_n be any sequence in D converging to $z_\infty \in D$. Since f and g are both continuous, $f(z_n) \rightarrow f(z_\infty)$ and $g(z_n) \rightarrow g(z_\infty)$.

Hence for each $\varepsilon > 0$, $\exists N_{\varepsilon, f}, N_{\varepsilon, g}$ so that $n > N_{\varepsilon, f} \Rightarrow |f(z_n) - f(z_\infty)| < \varepsilon/2$, and for $n > N_{\varepsilon, g} \Rightarrow |g(z_n) - g(z_\infty)| < \varepsilon/2$. Let $N_\varepsilon = \max\{N_{\varepsilon, f}, N_{\varepsilon, g}\}$.
If $n > N_\varepsilon$, then

$$\begin{aligned} & |f(z_n) + g(z_n) - (f(z_\infty) + g(z_\infty))| \\ &= |f(z_n) - f(z_\infty) + g(z_n) - g(z_\infty)| \\ &\leq |f(z_n) - f(z_\infty)| + |g(z_n) - g(z_\infty)| \\ &< \varepsilon/2 + \varepsilon/2 < \varepsilon. \end{aligned}$$

□

Thm

The composition of two continuous maps is continuous, when defined

Pf

Exercise



Math 424, Spring 2024

Lecture 07 - Some complex maps

In the last lecture we defined continuity of complex maps and saw that a couple of particularly simple maps (constant maps and the identity map) were continuous. We also saw that the sum of two continuous maps was continuous. We stated in a theorem that a product of continuous maps was continuous, but didn't prove it (which we'll do below), and left as an exercise showing that a composition of continuous maps was continuous.

In this lecture we'll use these tools to help us show some particularly important maps are continuous. First, though, we'll finish up the proof of a theorem from last time, and give one definition.

Thm

If $f, g: D \rightarrow \mathbb{C}$ are continuous maps, then their product $f \cdot g$ is also continuous.

Pf

Let $(z_n)_{n \in \mathbb{N}}$ be a sequence in D with limit $z_\infty \in D$. We must show $f(z_n) \cdot g(z_n) \rightarrow f(z_\infty) \cdot g(z_\infty)$.

To do this, we want to show for every $\varepsilon > 0$ we can choose an N so that if $n > N$,

$$|f(z_n)g(z_n) - f(z_\infty)g(z_\infty)| < \varepsilon.$$

Observe there is some N_1 so that if $n > N_1$, then $|f(z_n)| < 1 + |f(z_\infty)| + |g(z_\infty)|$. This follows from the fact that $f(z_n) \rightarrow f(z_\infty)$, and $|f(z_\infty)|$ and $|g(z_\infty)|$ are just constants. We write

$$\begin{aligned} |f(z_n)| &= |f(z_n) - f(z_\infty) + f(z_\infty)| \\ &\leq |f(z_n) - f(z_\infty)| + |f(z_\infty)| \\ &\leq |f(z_n) - f(z_\infty)| + |f(z_n)| + |g(z_n)| \quad (\star) \end{aligned}$$

As $f(z_n) \rightarrow f(z_\infty)$, there is an N_1 so $|f(z_n) - f(z_\infty)| < 1$ for $n > N_1$, and so the above inequality $(\star)$ holds for $n > N_1$ as well. Let's write k for $1 + |f(z_\infty)| + |g(z_\infty)|$.

Similarly, there are $N_2, N_3 \in \mathbb{N}$ so that $|f(z_n) - f(z_\infty)| < \frac{\varepsilon}{2k}$ if $n > N_2$, and $|g(z_n) - g(z_\infty)| < \frac{\varepsilon}{2k}$ if $n > N_3$. Now let $N = \max\{N_1, N_2, N_3\}$ and notice for $n > N$ we have

$$\begin{aligned} &|f(z_n)g(z_n) - f(z_\infty)g(z_\infty)| \\ &= |f(z_n)g(z_n) - f(z_n)g(z_\infty) + f(z_n)g(z_\infty) - f(z_\infty)g(z_\infty)| \\ &\leq |f(z_n)g(z_n) - f(z_n)g(z_\infty)| + |f(z_n)g(z_\infty) - f(z_\infty)g(z_\infty)| \end{aligned}$$

$$\begin{aligned}
&= |f(z_n)| |g(z_n) - g(z_\infty)| + |f(z_n) - f(z_\infty)| |g(z_\infty)| \\
&\leq (1 + |f(z_\infty)| + |g(z_\infty)|) |g(z_n) - g(z_\infty)| + |f(z_n) - f(z_\infty)| |g(z_\infty)| \\
&\leq (1 + |f(z_\infty)| + |g(z_\infty)|) |g(z_n) - g(z_\infty)| + |f(z_n) - f(z_\infty)| (1 + |f(z_\infty)| + |g(z_\infty)|) \\
&= (1 + |f(z_\infty)| + |g(z_\infty)|) (|g(z_n) - g(z_\infty)| + |f(z_n) - f(z_\infty)|) \\
&= k (|g(z_n) - g(z_\infty)| + |f(z_n) - f(z_\infty)|) \\
&\leq k \cdot \left(\frac{\varepsilon}{2k} + \frac{\varepsilon}{2k} \right) = k \cdot \frac{\varepsilon}{k} = \varepsilon.
\end{aligned}$$

□

Let's now make one definition very similar to a definition from calculus: we say a complex number $L \in \mathbb{C}$ is the limit of a complex map $f(z)$ as z approaches z_0 if for every open set $U \subseteq \mathbb{C}$ containing L there exists an open set $V \subseteq \mathbb{C}$ containing z_0 so that if $z \in V$, then $f(z) \in U$. If such an L exists, then we denote it by $L = \lim_{z \rightarrow z_0} f(z)$.

E.g., if $f(z)$ is the map $f(z) = \frac{3z^2 + 3}{z - i}$ we claim $\lim_{z \rightarrow i} f(z) = 6i$. Before verifying this, the following theorem will be helpful to have.

Thm

The limit $\lim_{z \rightarrow z_0} f(z)$ exists and equals L if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ so that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

Pf

Exercise

Rmk

The " $0 < |z - z_0| < \delta$ " in the theorem above effectively means we do not need to allow for $z = z_0$, as our function might not be defined at that point. This is the case, for example with $\lim_{z \rightarrow i} \frac{3z^2 + 3}{z - i}$.

Now to verify $\lim_{z \rightarrow i} \frac{3z^2 + 3}{z - i}$ equals $6i$. Notice this function equals $3(z+i)$ for all $z \neq i$, as $3(z+i)(z-i) = 3(z^2 + 1)$. Let $\varepsilon > 0$ be given and take $\delta = \varepsilon/3$.
 $|z - i| < \delta$, then we have

$$\begin{aligned} & |z - i| < \delta \\ \Rightarrow & |z - i| < \varepsilon/3 \\ \Rightarrow & 3|z - i| < \varepsilon \\ \Rightarrow & |3z - 3i| < \varepsilon \\ \Rightarrow & |3z + 3i - 6i| < \varepsilon \\ \Rightarrow & |3(z+i) - 6i| < \varepsilon \\ \Rightarrow & \left| \frac{3(z+i)(z-i)}{z-i} - 6i \right| < \varepsilon \\ \Rightarrow & \left| \frac{3z^2 + 3}{z-i} - 6i \right| < \varepsilon \end{aligned}$$

Using the ϵ - δ definition of the limit is tedious in practice, so we really like the following theorems as they make it much easier to compute limits, at least in some situations.

Thm

If $f: D \rightarrow \mathbb{C}$ is a continuous map defined on an open set $D \subseteq \mathbb{C}$, then for each point $z_0 \in D$,
 $\lim_{z \rightarrow z_0} f(z) = f(z_0)$.

Pf

Let U be any open set around $f(z_0)$. For any $z \in f^{-1}(U)$ we have $f(z) \in U$. □

Thm

Suppose $f_0: D \setminus \{z_0\} \rightarrow \mathbb{C}$ and $f: D \rightarrow \mathbb{C}$ are maps defined on open sets $D, D \setminus \{z_0\} \subseteq \mathbb{C}$ with the property $f_0(z) = f(z)$ for all $z \in D \setminus \{z_0\}$.

Then $\lim_{z \rightarrow z_0} f_0(z) = \lim_{z \rightarrow z_0} f(z)$.

Pf

Exercise

This theorem will allow us to "replace" a function such as $\frac{3z^2+1}{z-i}$ with $3(z+i)$ when computing the limit as z goes to i , for example. To make this as useful as possible, we'd like to know $3(z+i)$ was continuous, but we now have

enough tools at our disposal to prove that lots of functions are continuous.

Lemma

Every complex polynomial, $f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ is continuous.

Pf Polynomials are sums and products of constants and the identity maps. □

Thm

The map $z \mapsto \frac{1}{z}$ defined on $\mathbb{C} \setminus \{0\}$ is continuous.

Pf Note that $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ which in real and imaginary parts is $\frac{x}{x^2+y^2} + i \frac{-y}{x^2+y^2}$. These maps, $u(x,y) = \frac{x}{x^2+y^2}$ and $v(x,y) = \frac{-y}{x^2+y^2}$ are continuous as maps $\mathbb{R}^2 \setminus \{(0,0)\} \rightarrow \mathbb{R}$, and so by a homework problem, $u(x,y) + iv(x,y)$ is also continuous. □

Lemma

Every rational function $\frac{a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0}{b_m z^m + b_{m-1} z^{m-1} + \dots + b_1 z + b_0}$ is continuous on its domain ($\mathbb{C}$ without the roots of $b_m z^m + \dots + b_1 z + b_0$).

Pf Note $\frac{1}{b_m z^m + \dots + b_1 z + b_0}$ is the composition of a polynomial and $z \mapsto \frac{1}{z}$, so continuous. One

rational function is the product of the map with a polynomial. □

Now we can "easily" compute lots of limits:

Exercise

Compute $\lim_{z \rightarrow 1+i} \frac{z^2 - (1+2i)z + i-1}{z^2 - 2i}$ by first factoring the numerator and denominator.
(Hint: Check if $1+i$ is a root of each.)

Lecture 08 - Roots, logarithms, and branch cuts

We had previously seen that there were three distinct solutions to the equation $z^3 = 1$, given by $e^{i\frac{2\pi}{3} \cdot 0} = 1$, $e^{i\frac{2\pi}{3} \cdot 1}$, and $e^{i\frac{2\pi}{3} \cdot 2}$. More generally, for any non-zero complex number $w = \rho e^{i\theta}$ there are three solutions to the equation $z^3 = w$ given by $re^{i\theta/3}$, $re^{i(\theta/3 + \frac{2\pi}{3})}$, and $re^{i(\theta/3 + \frac{4\pi}{3})}$ where $r > 0$ is the unique non-negative real number so that $r^3 = \rho$.

For example, if $w = 8$, then $z^3 = 27$ is solved by 3 , $3e^{i\frac{2\pi}{3}}$, and $3e^{i\frac{4\pi}{3}}$; if $w = 2 + 2i = 8e^{i\pi/4}$, then $z^3 = w$ is solved by

$$2e^{i\frac{\pi}{12}}, 2e^{i[\frac{\pi}{12} + \frac{2\pi}{3}]} = 2e^{i\frac{3\pi}{4}}, 2e^{i[\frac{\pi}{12} + \frac{4\pi}{3}]} = 2e^{i\frac{17\pi}{12}}$$

as is easily verified:

$$(2e^{i\frac{\pi}{12}})^3 = 2^3 \cdot e^{i3\frac{\pi}{12}} = 8e^{i\pi/4}$$

$$(2e^{i\frac{3\pi}{4}})^3 = 2^3 \cdot e^{i3\frac{3\pi}{4}} = 8e^{i\frac{9\pi}{4}} = 8e^{i[\frac{\pi}{4} + \frac{8\pi}{4}]} = 8e^{i\pi/4}$$

$$(2e^{i\frac{17\pi}{12}})^3 = 2^3 \cdot e^{i3\frac{17\pi}{12}} = 8e^{i\frac{17\pi}{4}} = 8e^{i[\frac{\pi}{4} + \frac{16\pi}{4}]} = 8e^{i\pi/4}$$

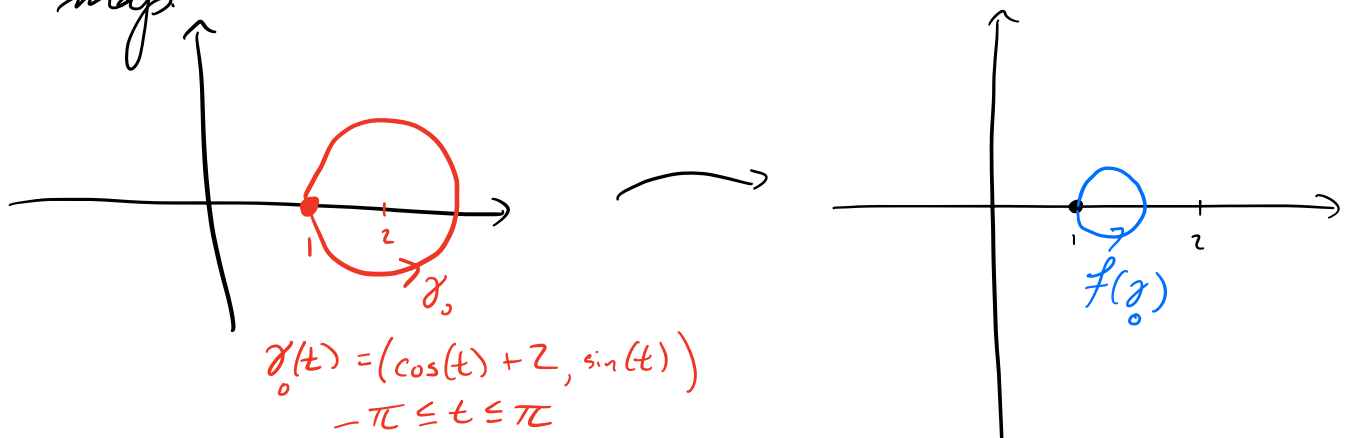
So, given that each (non-zero) complex number has

three cube roots, how should we define a cube root function, $\sqrt[3]{z}$ on $\mathbb{C}$?

One reasonable-seeming "guess" might be to just pick a choice of cube root and stick with it — this is basically what we do in calculus and algebra for square roots. We define the square root function on $[0, \infty)$ as the positive root.

What happens if we tried to do this on $\mathbb{C}$? We might, reasonably, want our cube root function to be continuous and agree with the cube root of the positive reals, so let's try to define a function $f(z)$ on $\mathbb{C}$ which is continuous and, say, has the property $f(1) = 1$ (as opposed to $f(1) = e^{i2\pi/3}$ or $f(1) = e^{i4\pi/3}$).

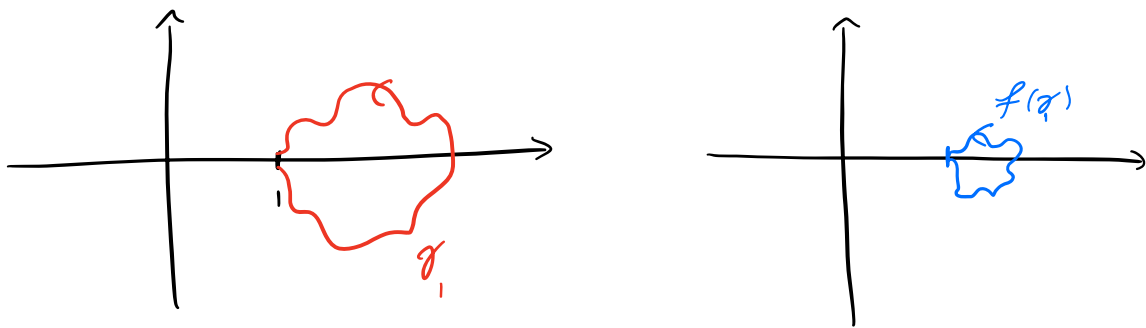
Now, let's consider what the image of a curve joining 1 to itself would be under this map.



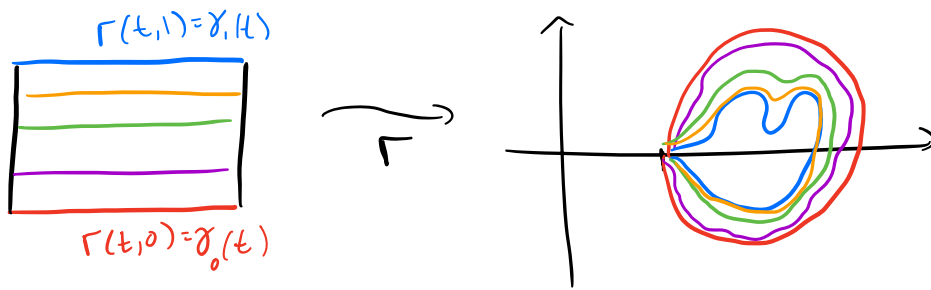
Notice our map is easiest to describe in polar coordinates as $re^{i\theta} \mapsto r^{1/3}e^{i\theta/3}$. In polar, our curve γ above is given by

$$\begin{aligned} r(t) &= \sqrt{[\cos(t)+2]^2 + [\sin(t)]^2} \\ &= \sqrt{\cos^2(t) + 4\cos(t) + 4 + \sin^2(t)} \\ &= \sqrt{5 + 4\cos(t)} \\ \theta(t) &= \tan^{-1}\left(\frac{\sin(t)}{\cos(t)+2}\right) \end{aligned}$$

If we "wiggle" our curve γ a little bit keeping the point at 1 fixed, we obtain another path joining 1 to itself when we apply γ

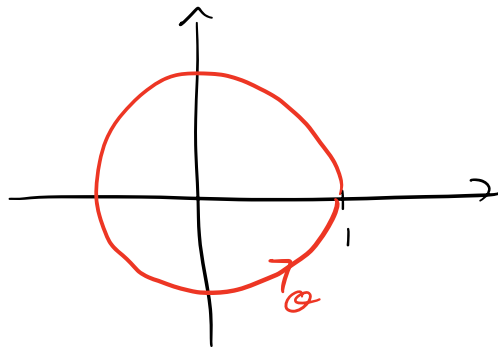


(Really, we should think of γ as a continuous map $\gamma: I \rightarrow \mathbb{C}$, where I is the unit interval $I = [0, 1]$. This "wiggling" really means we're continuously deforming γ . The way to make this precise is to consider a map $\Gamma: I \times I \rightarrow \mathbb{C}$ which is continuous, has the property that $\Gamma(t, 0) = \gamma_0(t)$, $\Gamma(t, 1) = \gamma_1(t)$, $\Gamma(0, s) = \Gamma(1, s) = 1$.



This map, Γ , is called a homotopy between γ_0 and γ_1 .

Now consider applying our putative cube root function to some other curve through 1, say the normal unit circle centered at 0.

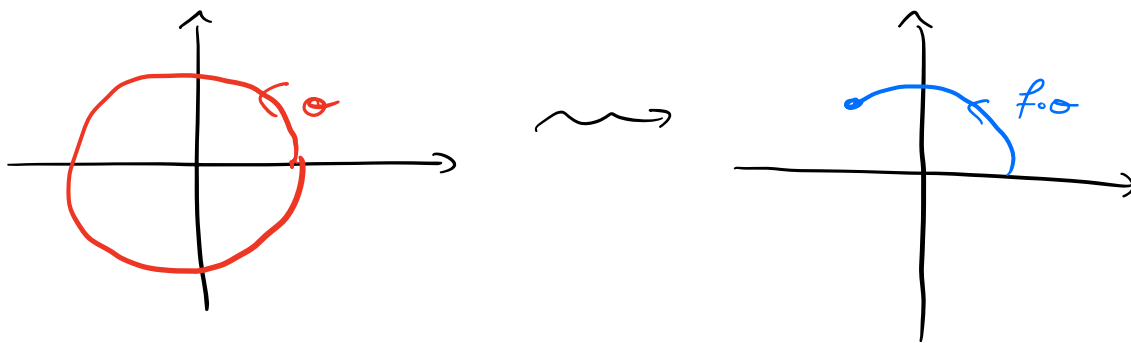


What should the image of this curve be? As we apply $f(re^{i\theta}) = r^{1/3} e^{i\theta/3}$ to points on this curve, walking around the circle, we're taking the cube root of $r=1$, so we still have distance 1 from the origin, but the angle $\frac{\theta}{3}$ on our image changes at $\frac{1}{3}$ the speed of θ .

$$\begin{aligned} \sigma(t) &= e^{i2\pi t} & 0 \leq t \leq 1 \\ f(\sigma(t)) &= e^{i\frac{2\pi t}{3}} \end{aligned}$$

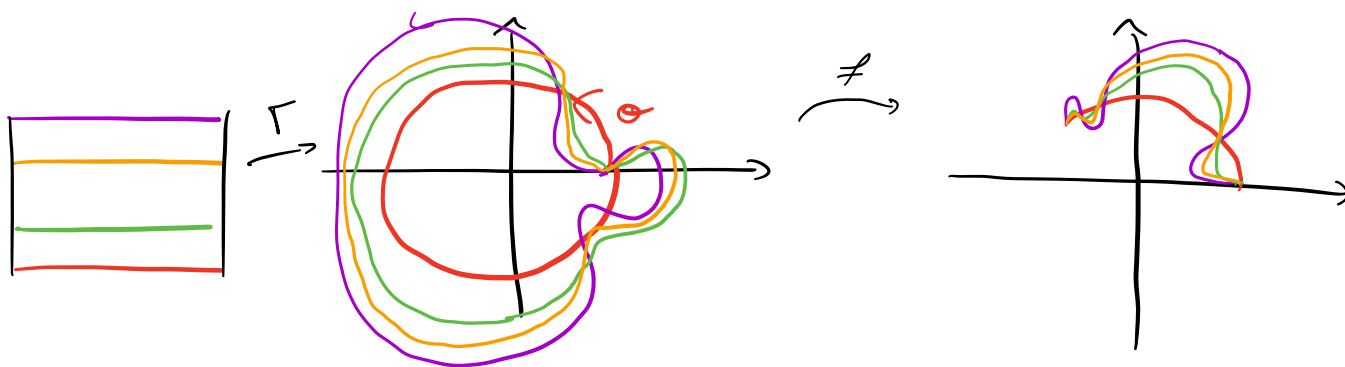
Thus by the time σ makes one full

revolution back to 1, the image curve $f \circ \sigma$ has only made one third of a revolution!



Notice in particular that the terminal point of the image curve $f \circ \sigma$ is $e^{i2\pi/3}$ — one of the "other" cube roots of 1!

Just as before, small (continuous) deformations of σ will yield deformations of $f \circ \sigma$, but which still give curves starting at 1 and ending at $e^{i2\pi/3}$.

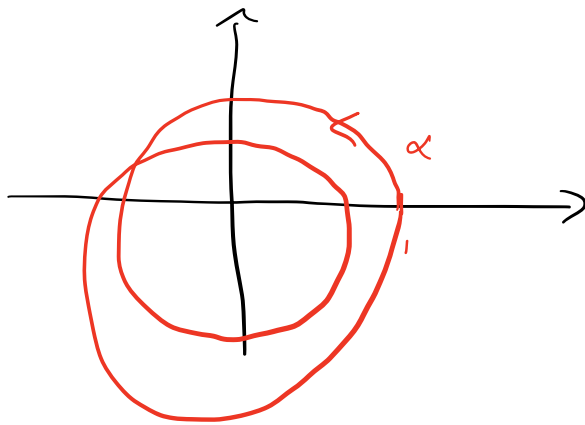


Let's observe that this means our map f is not a well-defined map! If it was, then every loop based at 1 would give a (deformed) loop based at 1 when f was

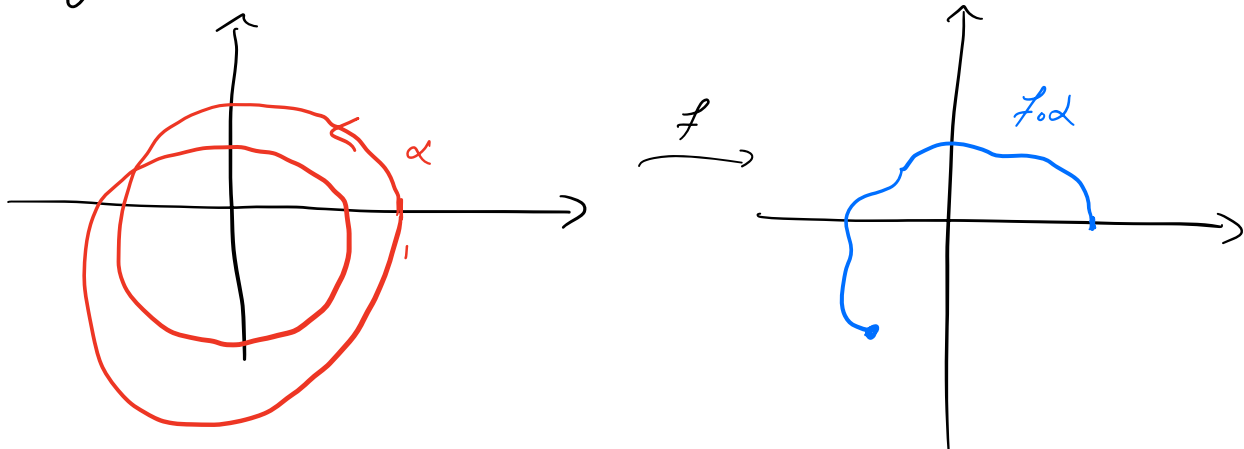
applied.

Before investigating this further, let's consider one more loop based at 1 and its image under our (not actually well-defined) map f .

Suppose α is the following loop:



If we apply $f(re^{i\theta}) = r^{1/3} e^{i\theta/3}$ to α , the angles θ change at $1/3$ the speed of θ , so if θ makes two revolutions around the origin, then our image curve should end at $e^{i4\pi/3}$ — which is our last remaining cube root of 1!



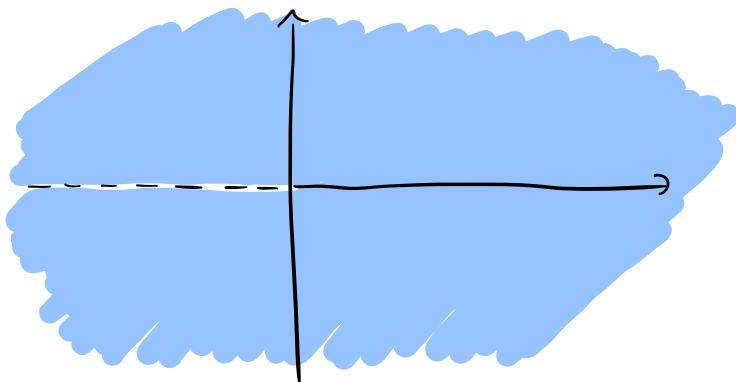
The observation to make is that we can't really define a cube root function on $\mathbb{C}$, at least not in the "obvious" way.

To try to make sense of a cube root function we will need to restrict ourselves to a subset of the plane. But what subset should we use?

We observed that if we continuously deformed a loop based at 1 a small enough amount, the terminal point of our loop should still be the same—deforming too much, though, and suddenly the terminal point "jumps" to one other cube root.

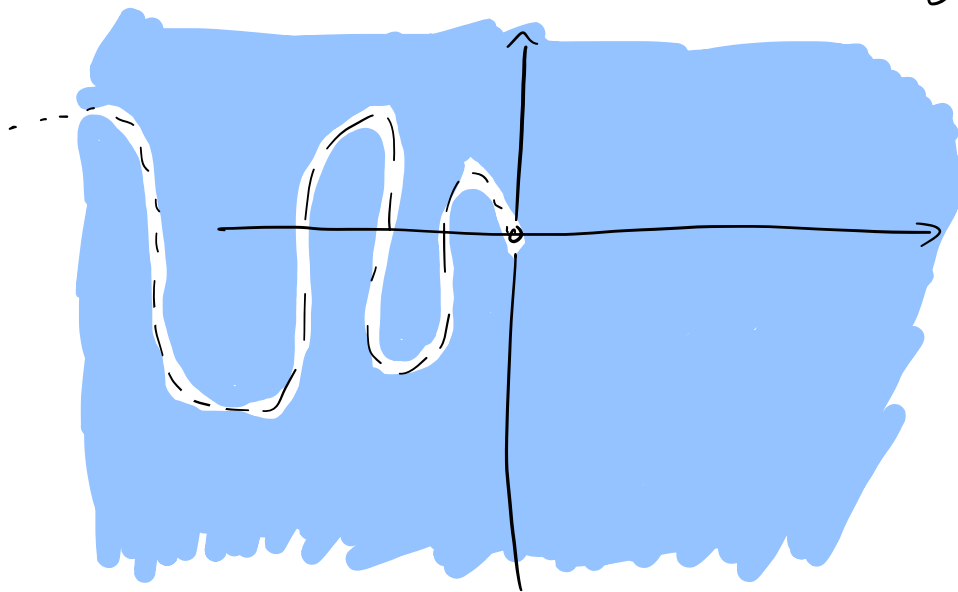
Notice that loops encircling the origin once gave us curves from 1 to $e^{i2\pi/3}$, and loops encircling the origin twice gave us curves from 1 to $e^{i4\pi/3}$. So, somehow (in a way we'll make precise later), encircling the origin seems to be the problem. If we're willing to take that on faith, then we should restrict ourselves to a subset of $\mathbb{C}$ where we can't have a loop encircling the origin.

One standard choice is to consider $\mathbb{C} \setminus \{z \in \mathbb{R} \mid z \leq 0\}$.



Removing the negative real axis prevents any curve from encircling the origin. Within this region our desired map $f(re^{i\theta}) = r^{1/3} e^{i\theta/3}$ becomes well-defined.

There's nothing super special about the negative real axis—you could just as easily remove some curve from 0 "to infinity" and still be left with another region where curves can not encircle the origin.



Making a choice of curve to remove from $\mathbb{C}$ so the origin is not encircled is called making a branch cut (for $z^{1/3}$ — other functions we might like to define to get a well-defined function might require ensuring we don't encircle some other point of $\mathbb{C}$).

After we make a branch cut, a choice of (well-defined) cube root in the remaining region is called a branch of the cube root.

Removing the negative real axis and then choosing continuous function mapping 1 to 1 (ie, $re^{i\theta} \mapsto r^{1/3} e^{i\theta/3}$) is often called the principal branch of the cube root.

Math 424, Spring 2024
Lecture 09 - Complex differentiability, Pt. 1

Recall from your first semester calculus class that the derivative of a function $f(x)$ at a point x_0 could be thought of as the slope of the line tangent to the graph $y = f(x)$ at the point $(x_0, f(x_0))$. A better way to think about the derivative is that tells us the "best" linear approximation of $f(x)$ at x_0 . This will be the point of view that will be most amenable for generalizing to complex-valued functions.

We'll first quickly recall how the derivative is defined in one variable, then generalize to two (real) variables, thinking of identifying $\mathbb{R}^2$ and $\mathbb{C}$ in the normal way, and finally see why being complex differentiable is not quite the same as being complex differentiable.

Recall that in first semester calculus, you defined the derivative of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ at some x_0 as

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided the limit exists. Suppose this limit does exist and equals some number m ,

$$\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = m$$

Let's rewrite this a little bit:

$$\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = m$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{m h}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0) - m h}{h} = 0$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0) - m h}{|h|} = 0$$

This last expression will be easier for us to generalize to higher dimensions. Let's notice, though, that multiplying a real number h by a fixed number m can be thought of as applying a linear transformation. Letting $T: \mathbb{R} \rightarrow \mathbb{R}$ be the linear transformation $T(h) = m h$, we have

$$\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0) - T(h)}{|h|} = 0$$

So, we can define the derivative of f at x_0 as the linear transformation $T: \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0) - T(h)}{|h|} = 0$$

Exercise

Show that if there is a T satisfying the above equation, it is unique. That is, if T' is another linear transformation satisfying the eqn above, then $T = T'$.

Now let's consider the case of a function $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Note this is really two functions together — say $F(x,y) = (u(x,y), v(x,y))$ where $u, v: \mathbb{R}^2 \rightarrow \mathbb{R}$. For example, if

$$F(x,y) = (x^2 + 3x - y^2, 2xy - 3y)$$

$x^2 + 3x$

then $u(x,y) = x^2 + 2x - y^2, v(x,y) = 2xy - 2y.$

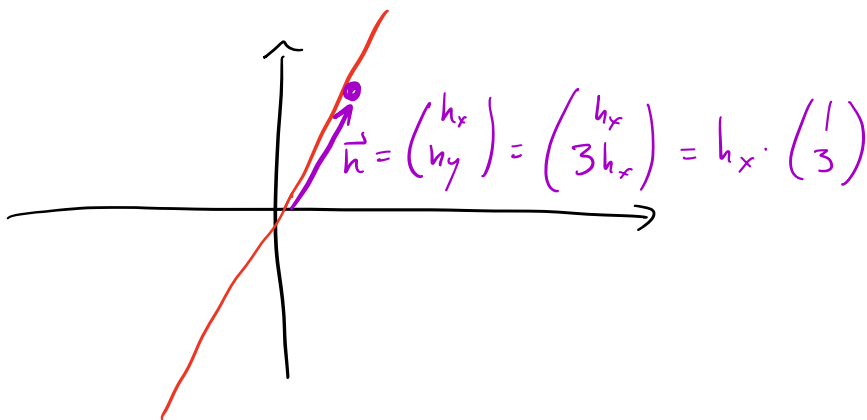
How should we make sense of the derivative of a function like this? Well, generalizing our modified definition of the derivative above, we might define the derivative of $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ to be the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfying

$$\lim_{\vec{h} \rightarrow 0} \frac{f(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) - T(\vec{h})}{\|\vec{h}\|} = 0$$

Notice for this limit to exist, we need to consider all directions of $\vec{h}$ approaching $\vec{0}$.

We can use this to determine the transformation T as a matrix.

First, let's establish some notation. We'll write our vectors as columns. Let's say $\vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is some fixed point, and write our $\vec{h}$ vectors as $\begin{pmatrix} h_x \\ h_y \end{pmatrix}$, keeping in mind we'll be taking limits as $\vec{h} \rightarrow 0$, meaning $h_x \rightarrow 0$ and $h_y \rightarrow 0$ — though they could be going to zero at different rates. E.g., we could have $h_y = 3h_x$, and so $\vec{h} \rightarrow 0$ would occur along the line of slope 3 through the origin.



Now suppose $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$,
so our limit as $\vec{h}$ goes to zero may be

written as

$$\lim_{\vec{h} \rightarrow 0} \frac{1}{\|\vec{h}\|} \cdot \begin{pmatrix} u(x_0+h_x, y_0+h_y) - u(x_0, y_0) - ah_x - bh_y \\ v(x_0+h_x, y_0+h_y) - v(x_0, y_0) - ch_x - dh_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If f is differentiable, this limit exists for all direction where $\vec{h}$ approaches $\vec{0}$. In particular, it would hold as $\vec{h}$ approaches $\vec{0}$ horizontally, and so our vectors become $\vec{h} = \begin{pmatrix} h_x \\ 0 \end{pmatrix}$ - i.e., $h_y = 0$. In that case our limit becomes

$$\lim_{h_x \rightarrow 0} \frac{1}{|h_x|} \cdot \begin{pmatrix} u(x_0+h_x, y_0) - u(x_0, y_0) - ah_x \\ v(x_0+h_x, y_0) - v(x_0, y_0) - ch_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which means we have

$$\lim_{h_x \rightarrow 0} \frac{u(x_0+h_x, y_0) - u(x_0, y_0) - ah_x}{|h_x|} = 0, \text{ and}$$

$$\lim_{h_x \rightarrow 0} \frac{v(x_0+h_x, y_0) - v(x_0, y_0) - ch_x}{|h_x|} = 0$$

But this precisely means $a = \frac{\partial u}{\partial x} \Big|_{(x_0, y_0)} = u_x(x_0, y_0)$,
and $c = \frac{\partial v}{\partial x} \Big|_{(x_0, y_0)} = v_x(x_0, y_0)$!

A similar calculation, with $\vec{h}$ approaching $\vec{0}$ vertically (i.e. $\vec{h} = \begin{pmatrix} 0 \\ h_y \end{pmatrix}$ and $h_x = 0$) will tell us $b = \frac{\partial u}{\partial y} \Big|_{(x_0, y_0)} = u_y(x_0, y_0)$ and $d = \frac{\partial v}{\partial y} \Big|_{(x_0, y_0)} = v_y(x_0, y_0)$.

Putting all of this together, our derivative T above (if it exists) is given by the matrix of partial derivatives,

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

This is called the Jacobian matrix of f .

It will turn out that the partial derivatives u_x, u_y, v_x, v_y can exist even if the function is not differentiable. If the function is differentiable, though, its derivative will be this matrix of partial derivatives. It will be the case that if all of the partial derivatives are continuous, then the map will be differentiable, however.

In the case of the map $f(x, y) = (x^2 + 3x - y^2, 2xy - 3y)$ our partial derivatives are

$$u_x = 2x + 3$$

$$v_x = 2y$$

$$u_y = -2y$$

$$v_y = 2x - 3$$

and these functions are of course continuous everywhere.

$$z^2 + 3z = (x + iy)^2 + 3(x + iy) = x^2 + 2ixy + y^2 + 3x + i3y$$

For the function $g(x, y) = (x^2 + 3x - y^2, 2xy + 3y)$

we have partial derivatives

$$u_x = 2x + 3$$

$$u_y = -2y$$

$$v_x = 2y$$

$$v_y = 2x + 3$$

The derivatives of our function f and g above are often denoted Df and Dg , and so we have

$$Df = \begin{pmatrix} 2x+3 & -2y \\ 2y & 2x-3 \end{pmatrix}$$

$$Dg = \begin{pmatrix} 2x+3 & -2y \\ 2y & 2x+3 \end{pmatrix}$$

These are almost the same, except for the ± 3 term in the second row, second column. As we'll see, this small difference is extremely important.

In our definition of the derivative $Df(\vec{x}_0)$ of a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ we had

$$\lim_{\vec{h} \rightarrow 0} \frac{f(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) - Df(\vec{x}_0)\vec{h}}{\|\vec{h}\|} = 0$$

we had to use $\|\vec{h}\|$ in the denominator, instead of $\vec{h}$, because we can't divide by a vector, and $Df(\vec{x}_0)$ was a linear transformation, which we think of as a 2×2 matrix. But if we think of the vector $\vec{x}_0 = (x_0, y_0)$ as the complex number $z_0 = x_0 + iy_0$,

and the vector-valued function $f(x,y) = (u(x,y), v(x,y))$ as being complex-valued, $f(x+iy) = u(x+iy) + iv(x+iy)$, then we can actually divide by a complex number $h = h_x + ih_y$ as $h \rightarrow 0$. Furthermore, just as we think of the derivative of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ as a single real number, maybe we should think of the derivative of a function $f: \mathbb{C} \rightarrow \mathbb{C}$ as a single complex number.

With all of this in mind, we define the complex derivative of a map $f: \mathbb{C} \rightarrow \mathbb{C}$ at a point z_0 as the complex number λ , if it exists, satisfying

$$\lim_{h \rightarrow 0} \frac{f(z_0+h) - f(z_0) - \lambda h}{h} = 0$$

As an example, consider the function g from above, which as a function $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ had the form $g(x,y) = (x^2 + 3x - y^2, 2xy + 3y)$.

Interpreting this as a complex function, let's write

$$g(x+iy) = x^2 + 3x - y^2 + i(2xy + 3y).$$

$$\begin{array}{l} z^2 + 3z \\ 2z + 3 \end{array}$$

We claim that $\lambda = 2i + 3$ will satisfy

our limit definition of the complex derivative of g at $z_0 = i$, which we now check:

$$\lim_{h \rightarrow 0} \frac{g(i+h) - g(i) - (2i+3)h}{h}$$

$$= \lim_{h \rightarrow 0} \frac{g(h_x + i(h_y+1)) - g(i) - (2i+3)(h_x + ih_y)}{h_x + ih_y}$$

$$= \lim_{h \rightarrow 0} \frac{h_x^2 + 3h_x - (h_y+1)^2 + i(2h_x(h_y+1) + 3(h_y+1)) - (-1+3i) - (3h_x - 2h_y + i(3h_y + 2h_x))}{h_x + ih_y}$$

$(3+2i)(h_x + ih_y) = 3h_x + i3h_y + i2h_x - 2h_y$

$$= \lim_{h \rightarrow 0} \frac{[h_x^2 + 3h_x - h_y^2 - 2h_y - 1 + 1 - 3h_x + 2h_y] + i[2h_x h_y + 2h_x + 3h_y + 3 - 3 - 3h_y - 2h_x]}{h_x + ih_y}$$

$$= \lim_{h \rightarrow 0} \frac{h_x^2 - h_y^2 + i 2h_x h_y}{h_x + ih_y}$$

$$= \lim_{h \rightarrow 0} \frac{h_x^2 + 2h_x h_y - h_y^2}{h_x + ih_y}$$

$$= \lim_{h \rightarrow 0} \frac{(h_x + ih_y)^2}{h_x + ih_y}$$

$$= \lim_{h \rightarrow 0} (h_x + ih_y) = \lim_{h \rightarrow 0} h = 0$$

So, $2i+3$ is the complex derivative of $g(x+iy) = x^2 + 3x - y^2 + i(2xy + 3y)$.

(We'll see where $2i+3$ came from soon.)

But what about our other function from earlier,

$$f(x,y) = (x^2 + 3x - y^2, 2xy - 3y)$$

As a complex-valued function we interpret this as

$$f(x+iy) = x^2 + 3x - y^2 + i(2xy - 3y)$$

Can any complex number $\lambda = a+ib$ satisfy the limit definition of the complex derivative of $f(x+iy)$ above at $z_0 = i$? Seeing whether any such $a+ib$ satisfies the limit definition "directly" can be difficult in general, but in the next lecture we'll see a tool (the Cauchy-Riemann equations) which will make it much easier to determine if there exists a value satisfying our complex derivative definition.

Math 424, Spring 2024

Lecture 10 - Complex differentiability, Part 2

In the last lecture we recalled that the derivative of a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

corresponded to the Jacobian matrix of f ,

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

We also saw that a complex function $f: \mathbb{C} \rightarrow \mathbb{C}$ may have a complex derivative f' which at a point z_0 we define to be

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

Since we can go back and forth between functions $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ on $\mathbb{R}^2$ and functions $x+iy \mapsto u(x,y) + iv(x,y)$ on $\mathbb{C}$, a natural question to consider is whether or not these two notions of derivative are the same or not.

Let's start with an example. Consider the map $f_{\mathbb{R}}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y \end{pmatrix}$, and

its complex counterpart, $f_{\mathbb{C}}: \mathbb{C} \rightarrow \mathbb{C}$ given by $z \mapsto \bar{z}$.

In both cases, if we write $f_{\mathbb{R}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ or $f_{\mathbb{C}}(x+iy) = u(x,y) + iv(x,y)$ we'll have $u(x,y) = x$ and $v(x,y) = -y$.

Notice the Jacobian matrix of $f_{\mathbb{R}}$ is

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and u, v are continuously differentiable, so $f_{\mathbb{R}}$ is differentiable.

Let's try to compute the complex derivative of $f_{\mathbb{C}}$ at some point $z_0 = x_0 + iy_0$. Using the limit definition this would be

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f_{\mathbb{C}}(z_0 + h) - f_{\mathbb{C}}(z_0)}{h} &= \lim_{h \rightarrow 0} \frac{f_{\mathbb{C}}(x_0 + iy_0 + h_x + ih_y) - (x_0 - iy_0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f_{\mathbb{C}}((x_0 + h_x) + i(y_0 + h_y)) - (x_0 - iy_0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{x_0 + h_x - i(y_0 + h_y) - x_0 + iy_0}{h} \end{aligned}$$

$$= \lim_{h \rightarrow 0} \frac{h_x - ih_y}{h} = \lim_{h \rightarrow 0} \frac{\overline{h}}{h}$$

If this limit exists, then for all sequences of complex numbers $(h_n)_{n \in \mathbb{N}}$ with $h_n \rightarrow 0$ we must have the same value for $\lim_{n \rightarrow \infty} \frac{\overline{h_n}}{h_n}$.

Consider first the sequence $h_n = \frac{1}{n}$. Then

$$\frac{\overline{h_n}}{h_n} = \frac{\overline{1/n}}{1/n} = 1$$

$$\text{so } \lim_{n \rightarrow \infty} \frac{\overline{h_n}}{h_n} = 1.$$

Now consider $h_n = \frac{i}{n}$. Then

$$\frac{\overline{h_n}}{h_n} = \frac{-i/n}{i/n} = -1$$

so for this sequence $\lim_{n \rightarrow \infty} \frac{\overline{h_n}}{h_n} = -1$.

Thus our map $f_{\mathbb{R}}$ is real differentiable, yet its complex counterpart $f_{\mathbb{C}}$ is not complex differentiable!

To understand this disconnect between real and complex differentiability, let's first recall that we define the derivative of a function f at a point P as the "best" linear approximation of f near P . When we are

thinking of real differentiability this means we consider 2×2 matrices, but for complex differentiability we have a single complex number. In both cases these objects act on points in the plane.

The 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $\begin{pmatrix} x \\ y \end{pmatrix}$ via matrix multiplication to give the point

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

The complex number $\zeta = \alpha + i\beta$ acts on the corresponding point $x + iy$ via complex multiplication yielding

$$\begin{aligned} (\alpha + i\beta)(x + iy) &= \alpha x + i\alpha y + i\beta x - \beta y \\ &= [\alpha x - \beta y] + i[\beta x + \alpha y] \end{aligned}$$

Now, when are these actions "the same"? Writing our vector $\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$ as the complex number $[ax + by] + i[cy + dx]$, we see multiplication by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ corresponds to multiplication by $\alpha + i\beta$ only when the equations

$$\begin{aligned} ax + by &= \alpha x - \beta y \\ cx + dy &= \beta x + \alpha y \end{aligned}$$

are satisfied for all x, y . Taking $x=1, y=0$ then gives

$$a = \alpha$$

$$c = \beta$$

and taking $x=0, y=1$ tells us

$$b = -\beta$$

$$d = \alpha$$

Putting all of this together, the 2×2 matrix which corresponds to multiplication by the complex number $\alpha + i\beta$ is

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

Writing $\alpha + i\beta$ as $re^{i\theta} = r\cos\theta + ir\sin\theta$, we see our matrix has the form

$$\begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

So, the matrix action corresponding to complex multiplication is (unsurprisingly) a rotation by θ composed with a scaling by r .

Relating all of this to differentiation, we have the following:

Theorem

A function $f_{\mathbb{R}}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ which is real differentiable at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ corresponds to a function $f_{\mathbb{C}}: \mathbb{C} \rightarrow \mathbb{C}$ given by $x+iy \mapsto u(x,y) + iv(x,y)$ which is complex differentiable at $x_0 + iy_0$ if and only if the Jacobian matrix

$$\begin{pmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{pmatrix}$$

satisfies $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$, in which case $f'_{\mathbb{C}}(x_0 + iy_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

Pf
Exercise

The equations above

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

are called the Cauchy-Riemann equations and they tell us precisely when a function $u(x,y) + iv(x,y)$

is complex differentiable.

If a complex map $f: U \rightarrow \mathbb{C}$ defined on an open subset $U \subseteq \mathbb{C}$ is complex differentiable at each point in its domain U , then we say f is holomorphic. Notice if the real and imaginary parts of f are u and v , then the theorem above tells us f is holomorphic iff u and v satisfy the Cauchy-Riemann equations at every point in U .

The Cauchy-Riemann equations make it relatively easy for us to check if a given function is complex differentiable or not without having to resort to the limit definition.

Example

Is the function $f(x+iy) = x^2 - 4x - y^2 + i(2xy + 4y)$ holomorphic? Here, $u(x,y) = x^2 - 4x - y^2$ and $v(x,y) = 2xy + 4y$. Notice $u_x = 2x - 4$ but $v_y = 2x + 4$. Thus $u_x \neq v_y$, and so f can not be holomorphic.

Example

Is the function $f(x+iy) = x^3 - 3xy^2 + 2x + i(3x^2y - y^3 + 2y)$ holomorphic. In this example, $u(x,y) = x^3 - 3xy^2 + 2x$ and $v(x,y) = 3x^2y - y^3 + 2y$. We compute the

following partial derivatives:

$$u_x = 3x^2 - 3y^2 + 2$$

$$u_y = -6xy$$

$$v_x = 6xy$$

$$v_y = 3x^2 - 3y^2 + 2$$

Note the Cauchy-Riemann equations are satisfied:
 $u_x = v_y$, $u_y = -v_x$. Thus the function is holomorphic.
Furthermore, its complex derivative is

$$\begin{aligned} f'(x+iy) &= u_x(x,y) + i v_x(x,y) \\ &= 3x^2 - 3y^2 + 2 + i 6xy \end{aligned}$$

Exercise

Express f and f' in the example above
as a function of z and $\bar{z}$ by writing
 $x = \frac{z + \bar{z}}{2}$ and $y = \frac{z - \bar{z}}{2i}$.

Not only are the Cauchy-Riemann equations useful
as a computational tool for determining if a given
 $f = u + iv$ is holomorphic, they are also helpful
for proving theorems

Thm

Suppose $f, g: U \rightarrow \mathbb{C}$ are two holomorphic

functions and let $\lambda \in \mathbb{C}$ be any fixed constant. Then the functions $f+g$, $f-g$, λf , $f \cdot g$, and $\frac{f}{g}$ are all holomorphic as well. (Notice though that the domain of $\frac{f}{g}$ is necessarily a subset of U as we may need to remove points $z \in U$ where $g(z)=0$.)

Pf
We will only prove that $f+g$ is holomorphic, but the proofs for the four functions are similar and left as exercises. Suppose

$$f(x+iy) = u(x,y) + iv(x,y), \text{ and } g(x+iy) = s(x,y) + it(x,y)$$

where, since f and g are assumed to be holomorphic, $u_x = v_y$, $u_y = -v_x$, $s_x = t_y$, and $s_y = -t_x$.

The function $f+g$ is given by

$$x+iy \mapsto [u(x,y) + s(x,y)] + i[v(x,y) + t(x,y)]$$

We must show $\frac{\partial}{\partial x}(u+s) = \frac{\partial}{\partial y}(v+t)$ and $\frac{\partial}{\partial y}(u+s) = -\frac{\partial}{\partial x}(v+t)$, but this is straightforward:

$$\begin{aligned} \frac{\partial}{\partial x}(u+s) &= u_x + s_x = v_y + t_y = \frac{\partial}{\partial y}(v+t) \\ \frac{\partial}{\partial y}(u+s) &= u_y + s_y = -v_x + (-t_x) = -\frac{\partial}{\partial x}(v+t). \end{aligned}$$

□

We can also show that compositions of holomorphic functions are holomorphic, but this will require the multivariable chain rule:

$$\frac{\partial}{\partial x} \varphi(\psi(x, y), \theta(x, y)) = \varphi_x(\psi(x, y), \theta(x, y)) \psi_x(x, y) + \varphi_y(\psi(x, y), \theta(x, y)) \theta_x(x, y)$$

$$\frac{\partial}{\partial y} \varphi(\psi(x, y), \theta(x, y)) = \varphi_x(\psi(x, y), \theta(x, y)) \psi_y(x, y) + \varphi_y(\psi(x, y), \theta(x, y)) \theta_y(x, y)$$

For notational convenience, ^{and similarly for the other partials} we may write $\frac{\partial \varphi}{\partial \psi}$ to mean $\varphi_x(\psi(x, y), \theta(x, y))$, then the multivariable chain rule reduces to

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial \psi} \cdot \frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

$$\frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial \psi} \cdot \frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

Thm

If $f: U \rightarrow \mathbb{C}$ and $g: V \rightarrow \mathbb{C}$ are holomorphic functions with $f(U) \subseteq V$, then $g \circ f: U \rightarrow \mathbb{C}$ is also holomorphic.

Pf

Suppose $f(x+iy) = u(x, y) + iv(x, y)$, $g(x+iy) = s(x, y) + it(x, y)$ where $u_x = v_y$, $u_y = -v_x$, $s_x = t_y$, $s_y = -t_x$. Observe the composition $g \circ f$ is given by

$$x+iy = s(u(x, y), v(x, y)) + it(u(x, y), v(x, y)).$$

We must show $\frac{\partial}{\partial x} s(u, v) = \frac{\partial}{\partial y} t(u, v)$ and $\frac{\partial}{\partial y} s(u, v) = -\frac{\partial}{\partial x} t(u, v)$. Using our "condensed" version of the multivariable chain rule above, we have

$$\begin{aligned}\frac{\partial}{\partial x} s(u, v) &= \frac{\partial s}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial s}{\partial v} \cdot \frac{\partial v}{\partial x} \\&= s_x(u, v) \cdot u_x + s_y(u, v) \cdot v_x \\&= t_y(u, v) \cdot v_y - t_x(u, v) \cdot (-u_y) \\&= t_x(u, v) \cdot u_y + t_y(u, v) \cdot v_y \\&= \frac{\partial t}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial t}{\partial v} \cdot \frac{\partial v}{\partial y} \\&= \frac{\partial}{\partial y} t(u, v)\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial y} s(u, v) &= \frac{\partial s}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial s}{\partial v} \cdot \frac{\partial v}{\partial y} \\&= s_x(u, v) \cdot u_y + s_y(u, v) \cdot v_y \\&= t_y(u, v) \cdot (-v_x) - t_x(u, v) \cdot u_x \\&= -(t_x(u, v) \cdot u_x + t_y(u, v) \cdot v_x) \\&= -\left(\frac{\partial t}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial t}{\partial v} \cdot \frac{\partial v}{\partial x}\right) \\&= -\frac{\partial}{\partial x} t(u, v).\end{aligned}$$

□

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Lecture 11 - Properties of Holomorphic Maps

In the last lecture we saw that holomorphic maps were "simply" maps which had a complex derivative at every point in their (open) domain. We noted that this is actually stronger than the corresponding map defined on $\mathbb{R}^2$ being differentiable in the sense of the total derivative & Jacobian matrix. The Cauchy-Riemann equations tell us that being real differentiable is prerequisite to being complex differentiable, but complex differentiability required an additional assumption, which essentially means the Jacobian matrix determines not just a real linear map, but a complex linear map.

In this lecture we'll start to see exactly how much stronger complex differentiability is than real differentiability by seeing some of the amazing properties we get "for free" once we know a map is holomorphic.

Let's first observe that if a function is complex differentiable, then its derivative is also complex differentiable. (Note this is not true for real functions - there are real functions which

are differentiable once, but not twice.)

To see this recall that if $f(x+iy) = u(x,y) + iv(x,y)$ is complex differentiable, then its complex derivative is given by

$$f'(x+iy) = u_x(x+iy) + iv_x(x,y).$$

To check if this function is itself complex differentiable, we check to see if the Cauchy-Riemann equations are satisfied. This means we need to verify the following system of PDE's holds:

$$\begin{aligned}\frac{\partial}{\partial x} u_x &= \frac{\partial}{\partial y} v_x \\ \frac{\partial}{\partial y} u_x &= -\frac{\partial}{\partial x} v_x\end{aligned}$$

To see $u_{xx} = v_{xy}$, let's note that $v_{xy} = v_{yx}$, and since f is complex differentiable, $v_y = u_x$, so

$$v_{xy} = v_{yx} = (v_y)_x = u_{xx}$$

Similarly,

$$-v_{xx} = -(v_x)_x = -(-u_y)_x = u_{yx} = u_{xy}$$

and so the 2nd equation is also satisfied. (Note

here we used the fact $u_y = -v_x$ which is supplied by the assumption f is complex differentiable).

The argument above essentially proves the following

Theorem

If $f: U \rightarrow \mathbb{C}$ defined on an open $U \subseteq \mathbb{C}$ is holomorphic, then it is infinitely differentiable, and each derivative $f^{(n)} := \frac{d^n}{dz^n} f$ is also holomorphic on U . □

This theorem will be extremely important when we discuss power series later.

Let's now give a geometric interpretation of holomorphicity. Recall that given two vectors $\vec{v}_1, \vec{v}_2$ in $\mathbb{R}^2$, the angle Θ between them can be computed as the value in $(-\pi, \pi]$ satisfying

$$\cos \Theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{\|\vec{v}_1\| \|\vec{v}_2\|}$$

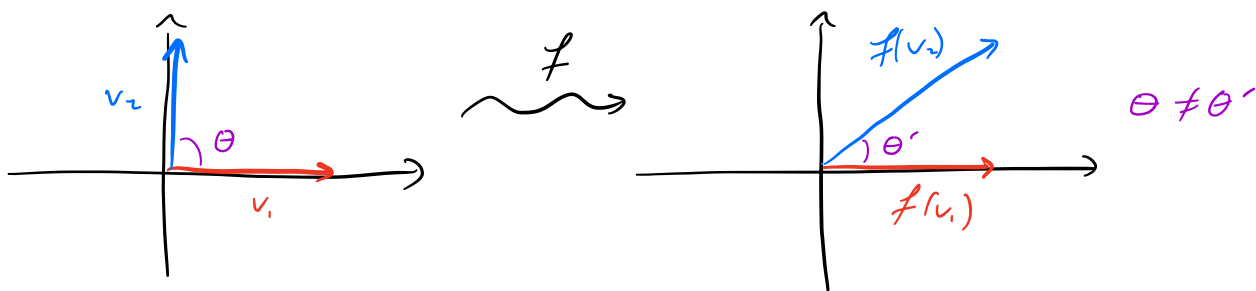
Interpreting $\vec{v}_1, \vec{v}_2$ as complex numbers,

$$\begin{aligned} v_1 &= x_1 + iy_1 & \text{if } \vec{v}_1 &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \\ v_2 &= x_2 + iy_2 & \text{if } \vec{v}_2 &= \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \end{aligned}$$

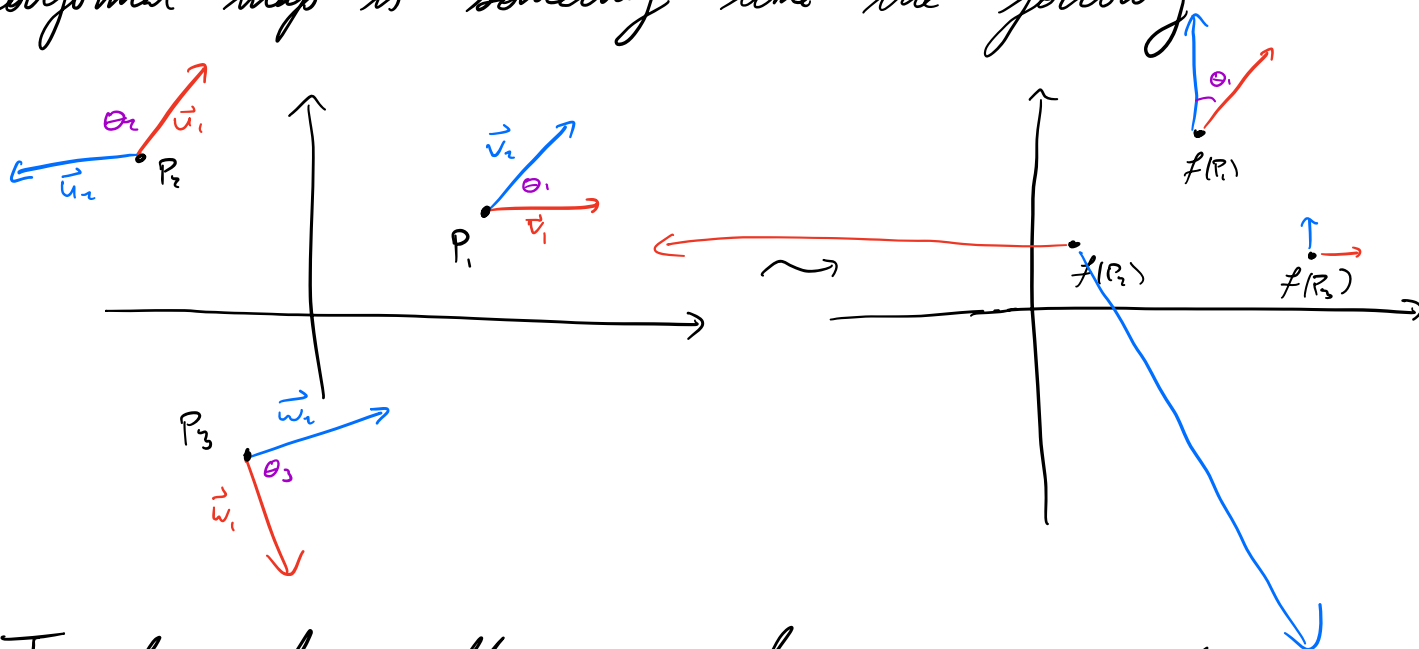
we define the angle between v_1 and v_2 as the real θ in $(-\pi, \pi]$ satisfying

$$\cos \theta = \frac{x_1 \cdot x_2 + y_1 \cdot y_2}{\sqrt{x_1^2 + y_1^2} \sqrt{x_2^2 + y_2^2}}$$

In general, maps $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ distort angles
 E.g., $f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y \end{pmatrix}$ shears horizontally and so changes the angle between vectors



A map $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is called conformal if it preserves angles at every point. Pictorially, a conformal map is something like the following



To be clear, the map here moves points

in the plane around, and in the process modifies vectors based at a point $P=(x_p, y_p)$ by sending the vector $\vec{v}_p = \begin{pmatrix} a \\ b \end{pmatrix}$ to the vector

$$\begin{pmatrix} u_x(x_p, y_p) & u_y(x_p, y_p) \\ v_x(x_p, y_p) & v_y(x_p, y_p) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a u_x(x_p, y_p) + b u_y(x_p, y_p) \\ a v_x(x_p, y_p) + b v_y(x_p, y_p) \end{pmatrix}$$

based at the point $f(P) = (u(x_p, y_p), v(x_p, y_p))$.

E.g., consider the map

$$Df = \begin{pmatrix} 2x+3 & -2y \\ 2y & 2x+3 \end{pmatrix}$$

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = \begin{pmatrix} x^2 + 3x - y^2 \\ 2xy + 3y \end{pmatrix}$$

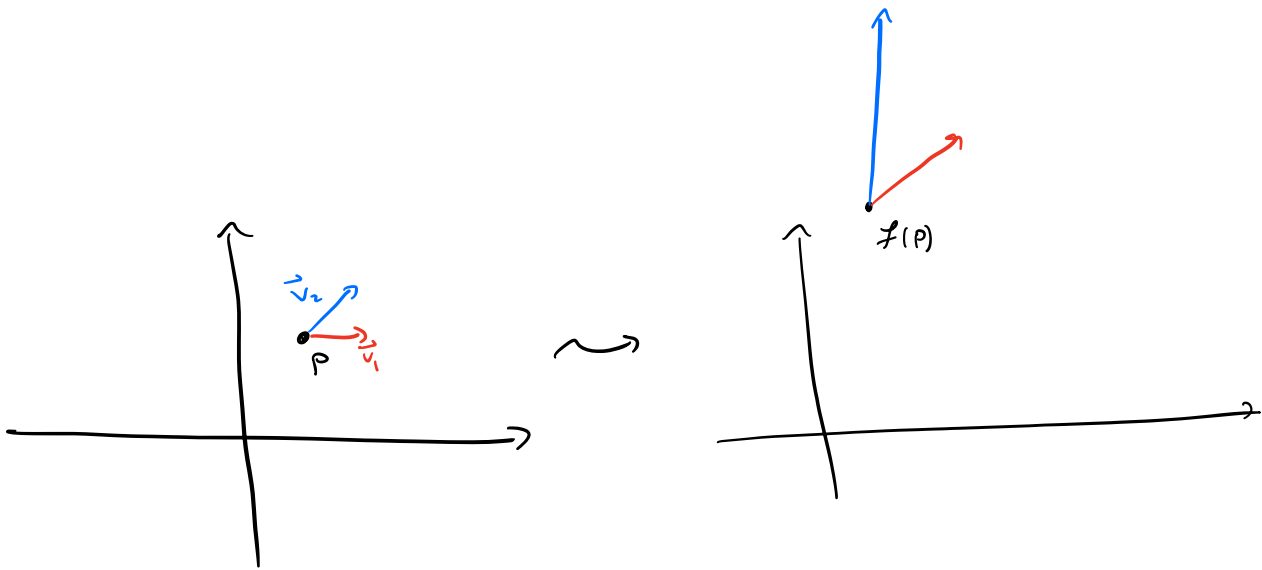
$$\begin{matrix} 4 + 9 - 9 \\ 12 + 9 = 21 \end{matrix}$$

Given vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ based at $P = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, they get sent to vectors

$$\begin{pmatrix} u_x(2,3) & u_y(2,3) \\ v_x(2,3) & v_y(2,3) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ 6 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} u_x(2,3) & u_y(2,3) \\ v_x(2,3) & v_y(2,3) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ 6 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 13 \end{pmatrix}$$

based at $f(P) = f\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 21 \end{pmatrix}$



The angles between the initial vectors is

$$\cos^{-1}\left(\frac{\vec{v}_1 \cdot \vec{v}_2}{\|\vec{v}_1\| \|\vec{v}_2\|}\right) = \cos^{-1}\left(\frac{1}{1 \cdot \sqrt{2}}\right) = \frac{\pi}{4}$$

and the angle b/w the image vectors is

$$\cos^{-1}\left(\frac{\begin{pmatrix} 7 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 13 \end{pmatrix}}{\left\|\begin{pmatrix} 7 \\ 6 \end{pmatrix}\right\| \cdot \left\|\begin{pmatrix} 1 \\ 13 \end{pmatrix}\right\|}\right) = \cos^{-1}\left(\frac{7+78}{\sqrt{49+36} \sqrt{1+169}}\right) = \cos^{-1}\left(\frac{85}{\sqrt{14,450}}\right) = \frac{\pi}{4}$$

In general, a map $f: U \rightarrow \mathbb{R}^2$ preserves angles between tangent vectors at a point $P \in U$ precisely when its derivative at P , Df_P , is a scalar multiple of a rotation matrix, a reflection, or a composition of such matrices.

For example,

$$\rightarrow \begin{pmatrix} \cos \frac{\pi}{3} & -\sin \frac{\pi}{3} \\ \sin \frac{\pi}{3} & \cos \frac{\pi}{3} \end{pmatrix} \cdot \begin{pmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \cos(-\pi/4) & -\sin(-\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{pmatrix}$$

$$= \begin{pmatrix} 36\sqrt{3} & -36 \\ -36 & -36\sqrt{3} \end{pmatrix}$$

is such a matrix.

In general, if a matrix (2×2) preserves all angles of all pairs of vectors, it must be a scalar multiple of a matrix that preserves the Euclidean dot product. Matrices which preserve the dot product are called orthogonal and are exactly the matrices whose transpose and inverse coincide:

$$\begin{aligned} \vec{u} \cdot \vec{v} &= (A\vec{u}) \cdot (A\vec{v}) \\ \Rightarrow \vec{u}^T \vec{v} &= (A\vec{u})^T (A\vec{v}) \\ &= \vec{u}^T A^T A \vec{v} \\ \Rightarrow \vec{u}^T \vec{v} - \vec{u}^T A^T A \vec{v} &= \vec{0} \\ \Rightarrow \vec{u}^T (\mathbf{I} - A^T A) \vec{v} &= \vec{0} \end{aligned}$$

Let's momentarily write $U = \mathbf{I} - A^T A$. So, $\vec{u}^T U \vec{v} = 0$ for all $\vec{u}, \vec{v}$. In particular, if $\vec{u} = \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ this gives

$$\begin{aligned} (1 \ 0) \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \Rightarrow (1 \ 0) \begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

$$\Rightarrow u_{11} = 0.$$

Other choices of $\bar{u}, \bar{v}$ will show all other entries of U are zero, so $U = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ meaning $I - A^T A = 0$, so $A^T A = I$.

Let's observe that if $f(x+iy) = u(x,y) + iv(x,y)$ satisfies the Cauchy-Riemann equations, its total derivative is a multiple of an orthogonal matrix. We had previously observed if $f'(z_0) = re^{i\theta}$, the corresponding total derivative is

$$r \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

It's easy to see $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ is orthogonal:

$$\begin{aligned} & \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}^T \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2 \theta + \sin^2 \theta & -\cos \theta \sin \theta + \cos \theta \sin \theta \\ -\cos \theta \sin \theta + \cos \theta \sin \theta & \cos^2 \theta + \sin^2 \theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Thus we have justified that holomorphic maps are conformal.

Let's end this lecture by mentioning an important connection between holomorphic functions and (real) differential equations.

One of the most common PDE's in application is the Laplace equation. In two variables this is the PDE

$$u_{xx} + u_{yy} = 0$$

where $u: U \rightarrow \mathbb{R}$ is some function defined on an open set $U \subseteq \mathbb{R}^2$. This is also sometimes written $\nabla^2 u = 0$ (here ∇ is the gradient operator, $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ and ∇^2 is short-hand for ∇ dotted with itself $\nabla \cdot \nabla u = \nabla \cdot (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \cdot (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = u_{xx} + u_{yy}$) or $\Delta u = 0$ (Δ is the Laplace operator, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \nabla \cdot \nabla$). This PDE $\Delta u = 0$, arises in many physical applications, where some vector field is proportional to the gradient of a function (in electrostatics this gives Ohm's law of electrical conductance; in chemistry this gives Fick's law of diffusion). In some instances the divergence of that vector field, $\text{div}(\vec{F}) = \nabla \cdot \vec{F}$, is zero.

When $\vec{F} = \lambda \nabla u$, $\text{div}(\vec{F}) = 0$ is precisely Laplace's equation.

A function u satisfying Laplace's equation, $u_{xx} + u_{yy} = 0$, is called harmonic. Let's notice that if $f: U \rightarrow \mathbb{C}$, $U \subseteq \mathbb{C}$ open, is holomorphic, then its real and imaginary parts are necessarily harmonic. To see this, suppose $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic, so the Cauchy-Riemann equations hold: $u_x = v_y$, $u_y = -v_x$. Now we simply compute

$$u_{xx} + u_{yy} = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = v_{yx} - v_{yx} = 0$$

$$v_{xx} + v_{yy} = (-u_y)_x + (u_x)_y = -u_{yx} + u_{xy} = -u_{xy} + u_{xy} = 0$$

This will mean that many nice properties of harmonic functions (to discuss later) carry over to holomorphic functions

Notice that a consequence of this fact that $f = u + iv$ being holomorphic implies u, v harmonic is that if a given u is not harmonic it can not be the real part of a holomorphic function. E.g., $u(x,y) = x^2 + y^2$ will have

$$u_{xx} + u_{yy} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2y) = 2 + 2 = 4$$

Thus there is no holomorphic $f(x+iy) = u(x,y) + iv(x,y)$ where $u(x,y) = x^2 + y^2$!

But now a different question arises: given a harmonic $u: U \rightarrow \mathbb{R}$, does there exist a harmonic $v: U \rightarrow \mathbb{R}$ so $u+iv$ is holomorphic?

Given a harmonic $u: U \rightarrow \mathbb{R}$, a harmonic $v: U \rightarrow \mathbb{R}$ so that $u+iv$ is holomorphic is called a harmonic conjugate of u . Before discussing whether harmonic conjugates necessarily exist, let's mention that harmonic conjugates (if they exist) are (almost) unique

Thm

Suppose $u: U \rightarrow \mathbb{R}$ is a harmonic function

and $v_1, v_2: U \rightarrow \mathbb{R}$ are two harmonic conjugates of u . Then $v_1 = v_2 + k$ for some constant k .

Pf

Note $f_1 = u + iv_1$ and $f_2 = u + iv_2$ are holomorphic.

Thus $f_1 - f_2$ is holomorphic, but

$f_1 - f_2 = (u + iv_1) - (u + iv_2) = i(v_1 - v_2)$. Thus

$i(v_1 - v_2)$ is holomorphic, but by a homework exercise, the only purely imaginary-valued holomorphic functions are constants. So $i(v_1 - v_2) = ik$ and $v_1 - v_2 = k$, so $v_1 = v_2 + k$. □

Before seeing the general construction of harmonic conjugates, let's work through a concrete example. Consider $u(x, y) = xy$. Notice $u_{xx} + u_{yy} = \frac{\partial^2}{\partial x^2} y + \frac{\partial^2}{\partial y^2} x = 0 + 0 = 0$, so u is harmonic. We want to find a v so $u + iv$ is holomorphic, which means the Cauchy-Riemann equations must be satisfied. Thus v needs to satisfy

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

But we know what u_x and u_y are:

$$\begin{aligned} v_y &= y \\ v_x &= -x \end{aligned}$$

Let's integrate to find v :

$$v = \int \frac{\partial v}{\partial y} dy = \int y dy = \frac{y^2}{2} + h(x)$$

But $v_x = -x$, so

$$\frac{\partial}{\partial x} \left(\frac{y^2}{2} + h(x) \right) = -x$$

$$\Rightarrow h'(x) = -x$$

$$\Rightarrow h(x) = \int -x dx = -\frac{x^2}{2} + C$$

So we may take $v(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + C$
Let's verify this is harmonic:

$$v_x = -x$$

$$v_y = y$$

$$v_{xx} = -1$$

$$v_{yy} = 1$$

Thus $v_{xx} + v_{yy} = -1 + 1 = 0$, so v is harmonic.

To verify that the map

$$f(x+iy) = u(x, y) + i v(x, y)$$

$$= xy + i \left(\frac{y^2}{2} - \frac{x^2}{2} + C \right)$$

is holomorphic, we check the Cauchy-Riemann equation,

$$u_x = y = v_y$$

$$u_y = x = -(-x) = -v_x$$

Math 424, Spring 2024

Lecture 12 - Harmonic conjugates, complex power series

In the last lecture we saw that if $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic, then the functions u and v must be harmonic - i.e.,

$$u_{xx} + u_{yy} = 0; \text{ and}$$

$$v_{xx} + v_{yy} = 0$$

Given a function $u: U \rightarrow \mathbb{R}$, where $U \subseteq \mathbb{R}^2$ is open, if u is harmonic, a natural question consider is if there is a choice of $v: U \rightarrow \mathbb{R}$ so $x+iy \mapsto u(x,y) + iv(x,y)$ is holomorphic. Such a v , if it exists, is called a "harmonic conjugate" of u .

So, now the question becomes do harmonic conjugates necessarily exist, and if so, how do we compute them?

Supposing $u: U \rightarrow \mathbb{R}$ is harmonic and we seek a $v: U \rightarrow \mathbb{R}$ so $u+iv$ is holomorphic, we must find v so the Cauchy-Riemann equations are satisfied. As u is some known function, its partial derivatives u_x and u_y are known, so we must find

a v with $v_x = -u_y$ and $v_y = u_x$. So, we know what we want the partial derivatives of v to be, and the "obvious" thing to do would be to integrate these partials. I.e., v should be

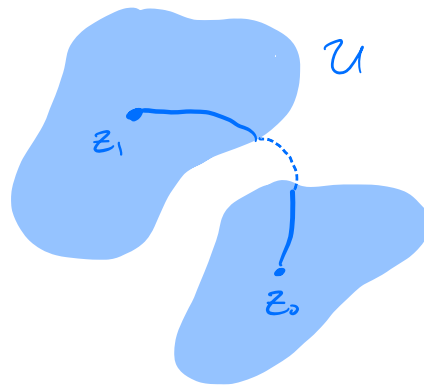
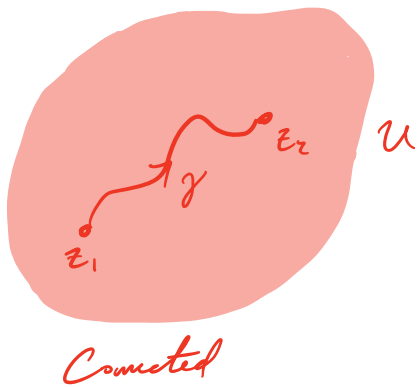
$$\begin{aligned} v &= \int \frac{\partial v}{\partial y} dy = \int \frac{\partial v}{\partial x} dx \\ &= \int u_x dy = \int -u_y dx \quad (\text{by CR-eqn's}) \end{aligned}$$

Exercise

- Verify that $u = \ln(x^2 + y^2)$ is harmonic on its domain $U = \mathbb{R}^2 \setminus \{(0,0)\}$.
- Compute its harmonic conjugate v .
- Is there anything "wrong" with v ?
 - Should get $v = 2\arctan(\frac{y}{x}) + C$ and can verify $f(x+iy) = \ln(x^2+y^2) + i2\arctan(\frac{y}{x})$ is holomorphic on $\mathbb{C} \setminus \{z \mid \operatorname{Im}(z) = 0\}$.
 - Issue: $v = 2\arctan(\frac{y}{x})$ has a smaller domain than $u = \ln(x^2+y^2)$.

In general, a harmonic conjugate to a harmonic $u: U \rightarrow \mathbb{R}$ is only guaranteed to exist on the entire domain U when U satisfies a topological condition called "simple connectedness".

We say a subset $U \subseteq \mathbb{C}$ is connected if for every pair of points $z_0, z_1 \in U$ there is a path $\gamma: [0, 1] \rightarrow U$ so $\gamma(0) = z_0, \gamma(1) = z_1$.

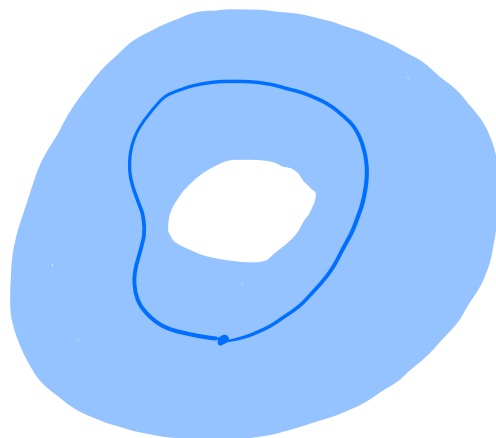


Not connected b/c a path from z_0 to z_1 would exit U

A set U is simply connected if it is connected and moreover all closed paths within U can be continuously deformed to a point. ^{stays in U}
Intuitively, simply connected subsets don't have any "holes" — the "holes" are what may prevent us from shrinking a closed path to a point.



Simply connected

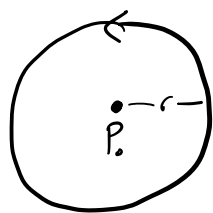


Connected, but not simply connected

Proving every harmonic function $u:U \rightarrow \mathbb{C}$ has a harmonic conjugate $v:U \rightarrow \mathbb{C}$ if and only if U is simply connected will be easiest after we've seen how to integrate holomorphic functions, so we'll postpone that proof until later.

We can go ahead and note, though, that since open sets are unions of open balls, and since every open ball is simply connected, every harmonic function u does have a local harmonic conjugate. That is, if $u:U \rightarrow \mathbb{R}$ is a harmonic function defined on an open set U , then for every point P_0 in U there exists some $\varepsilon > 0$ so that u has a harmonic conjugate v defined in $B_\varepsilon(P_0)$.

One of the marvelous properties of harmonic functions is that the value of a harmonic function at any point equals the average of that function over any circle centered at that point (contained in U).



$$u(P_0) = \frac{1}{2\pi r} \int_{\partial B_r(P_0)} u(x,y) ds$$

Recall from multivar. calculus that the line integral $\int f(x,y) ds$ where C is a curve parametrized by $(x(t), y(t))$ for $a \leq t \leq b$ is

$$\int_C f(x,y) ds = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt$$

For our ball of radius r around $P_0 = (x_0, y_0)$, we parametrize using $(x_0 + r \cos(t), y_0 + r \sin(t))$ $0 \leq t \leq 2\pi$, and

$$\begin{aligned} \frac{1}{2\pi r} \int_{\partial B_r(P_0)} u(x,y) ds &= \frac{1}{2\pi r} \int_0^{2\pi} u(x_0 + r \cos t, y_0 + r \sin t) \sqrt{r^2 \sin^2 t + r^2 \cos^2 t} dt \\ &= \frac{1}{2\pi r} \int_0^{2\pi} u(x_0 + r \cos t, y_0 + r \sin t) \cdot r dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r \cos t, y_0 + r \sin t) dt \end{aligned}$$

Thinking of this quantity as a function of the radius r , we first show this expression is constant by showing its derivative with respect to r is zero:

$$\begin{aligned} \frac{d}{dr} \frac{1}{2\pi r} \int_{\partial B_r(P_0)} u(x,y) ds &= \frac{d}{dr} \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r \cos t, y_0 + r \sin t) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{d}{dr} u(x_0 + r \cos t, y_0 + r \sin t) dt \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi} \int_0^{2\pi} \left[u_x(x_0 + r \cos t, y_0 + r \sin t) \cdot \cos(t) + \right. \\
&\quad \left. u_y(x_0 + r \cos t, y_0 + r \sin t) \cdot \sin(t) \right] dt \\
&= \frac{r}{2\pi r} \int_0^{2\pi} (u_x(x(t), y(t)) \cos(t) + u_y(x(t), y(t)) \sin(t)) dt \\
&= \frac{1}{2\pi r} \int_0^{2\pi} (u_x(x(t), y(t)) r \cos(t) + u_y(x(t), y(t)) r \sin(t)) dt \\
&= \frac{1}{2\pi r} \int_{\partial B_r(P_0)} u_x dy - u_y dx
\end{aligned}$$

where here we used the multivariable chain rule and $x(t) = x_0 + r \cos(t)$, $y(t) = y_0 + r \sin(t)$ to compute $dx = -r \sin(t) dt$, $dy = r \cos(t) dt$.

Recall now that Green's theorem states

$$\int_{\partial C} P dx + Q dy = \int_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Applying Green's theorem with $P = -u_y$ and $Q = u_x$, we then have

$$\begin{aligned}
\frac{d}{dr} \frac{1}{2\pi r} \int_{\partial B_r(P_0)} u(x, y) ds &= \frac{1}{2\pi r} \int_{\partial B_r(P_0)} -u_y dx + u_x dy \\
&= \frac{1}{2\pi r} \int_{B_r(P_0)} \left(\frac{\partial}{\partial x} u_x - \left(\frac{\partial}{\partial y} - u_y \right) \right) dA
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi r} \int_{B_r(P_0)} (u_{xx} + u_{yy}) dA \\
&= 0.
\end{aligned}$$

Thus the average of u over a circle centered at P_0 is independent of the radius r — all radii give the same average. Now consider the difference between the average and $u(P_0)$:

$$\begin{aligned}
&\left| \frac{1}{2\pi r} \int_{\partial B_r(P_0)} u(x, y) ds - u(x_0, y_0) \right| \\
&= \left| \frac{1}{2\pi r} \int_{\partial B_r(P_0)} u(x, y) ds - \frac{1}{2\pi r} \cdot 2\pi r u(x_0, y_0) \right| \\
&= \left| \frac{1}{2\pi r} \int_{\partial B_r(P_0)} u(x, y) ds - \frac{1}{2\pi r} \int_{\partial B_r(P_0)} u(x_0, y_0) ds \right| \\
&= \frac{1}{2\pi r} \left| \int_{\partial B_r(P_0)} (u(x, y) - u(x_0, y_0)) ds \right| \\
&= \frac{1}{2\pi r} \left| \int_0^{2\pi} [u(x(t), y(t)) - u(x_0, y_0)] r dt \right| \\
&= \frac{1}{2\pi} \left| \int_0^{2\pi} [u(x(t), y(t)) - u(x_0, y_0)] dt \right|
\end{aligned}$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |u(x(t), y(t)) - u(x_0, y_0)| dt$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} M_r dt = \frac{1}{2\pi} 2\pi M_r = M_r$$

where M_r is the maximum of $|u(x(t), y(t)) - u(x_0, y_0)|$ over the interval $[0, 2\pi]$, $(x(t), y(t)) = (x_0 + r \cos t, y_0 + r \sin t)$. Notice now that since u is continuous,

$$\lim_{r \rightarrow 0} u(x_0 + r \cos t, y_0 + r \sin t) = u(x_0, y_0)$$

and so as r goes to zero, M_r goes to zero. Hence

$$\left| \frac{1}{2\pi r} \int_{\partial B_r(P)} u(x, y) ds - u(x_0, y_0) \right| = 0$$

$$\text{So } u(x_0, y_0) = \frac{1}{2\pi r} \int_{\partial B_r(P)} u(x, y) ds$$

We have thus proven the following

Thm (Mean Value Property of Harmonic Functions)

Let $u: U \rightarrow \mathbb{R}$ be a harmonic function defined on an open set U . For every $P \in U$ and every $r > 0$ so that $B_r(P) \subseteq U$, the value of u at P

equals the average of u over the circle of radius r centered at P .

□

We can take advantage of the mean value property to show that a harmonic function $u: U \rightarrow \mathbb{R}$, $U \subseteq \mathbb{R}^2$ open and connected, does not achieve a global maximum unless u is constant.

Thm (Maximum Principle for Harmonic Functions)

Suppose $u: U \rightarrow \mathbb{R}$ is a harmonic function defined on a connected, open subset $U \subseteq \mathbb{R}^2$. If there exists a $M \in \mathbb{R}$ and a point $P_0 \in U$ so that $u(P_0) = M$ and $u(P) \leq M$ for all $P \in U$, then u is constant.

Pf

We will show $u(P) = M$ for all $P \in U$.
Fix an arbitrary $P \in U$.

Consider any open ball $B_r(P_0)$ around P_0 which remains in U . By the mean value property,

$$\frac{1}{2\pi r} \int_{\partial B_r(P_0)} u(x, y) ds = u(P_0) = M$$

But by assumption, $u(x, y) \leq M$ for all $(x, y) \in \partial B_r(P_0)$. Thus

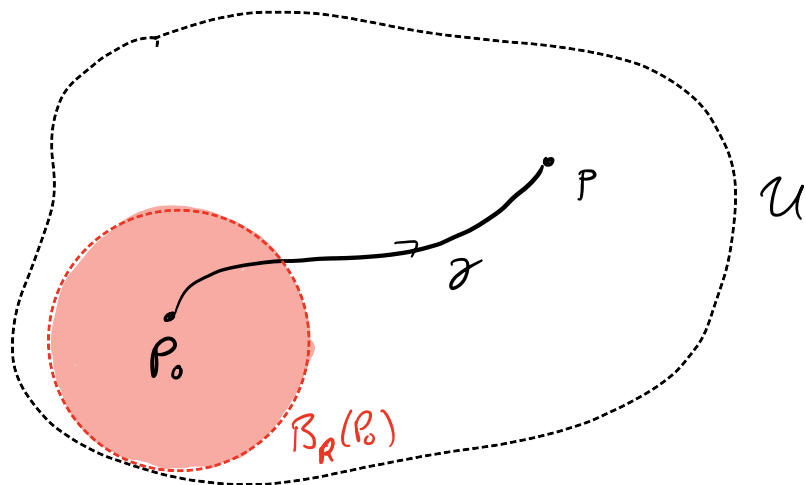
$$\frac{1}{2\pi r} \int_{\partial B_r(P_0)} u(x, y) ds \leq \frac{1}{2\pi r} \int_{\partial B_r(P_0)} M ds \leq \frac{1}{2\pi r} 2\pi r M = M$$

As u is continuous, equality only occurs if $u(x,y) = M$ for all $(x,y) \in \partial B_r(P_0)$. But this holds for all r so that $B_r(P_0) \subseteq U$.

If $U = \mathbb{C}$, then for every $P \in \mathbb{C}$ there is an r so $P \in B_r(P_0)$, so every $P \in \mathbb{C}$ has $u(P) = M$ and the proof is complete.

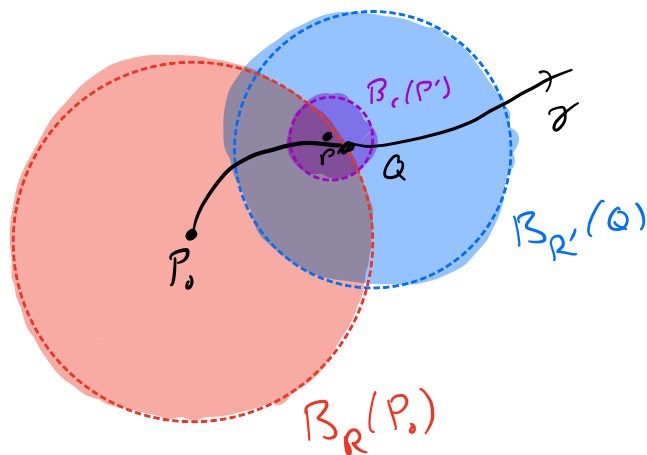
If $U \neq \mathbb{C}$, let R be the largest value so $B_R(P_0) \subseteq U$. If $P \in B_R(P_0)$, then $u(P) = M$ and we're done. If $P \notin B_R(P_0)$, let γ be any path from P_0 to P which remains in U .

We will show each point on γ admits a ball around that point where u equals M for each point in the ball.



Pick a point Q on $\gamma \cap \partial B_R(P_0)$. Since Q is a boundary point of $B_R(P_0)$, every open ball $B_r(Q)$ around Q intersects $B_R(P_0)$. Choose

some ball (you may as well choose the one which is as large as possible but stays in U) around Q , call it $B_{R'}(Q)$. Let $P' \in B_R(P_0) \cap B_{R'}(Q)$. Because $P' \in B_R(P_0)$, $u(P') = M$. Notice there is a radius r so that $B_r(P')$ contains Q



But by the mean value property, this means $u(Q) = M$, and also $u(S) = M$ for each $S \in B_{R'}(Q)$. Continuing in this way we can show every point T on γ has $u(T) = M$, so in particular $u(P) = M$. □

Math 424, Spring 2024
Lecture 13 - Power Series

Let's turn away from harmonic functions for a moment to consider holomorphic function again. In particular, we want to see the relationship between functions that can be represented as (complex) power series and holomorphicity.

The outline of our power series discussion will be as follows:

- Discuss convergence of power series
- Show functions defined as (convergent) power series are necessarily holomorphic.
- Show every holomorphic function can locally be represented as a power series.

In general, a power series of a complex variable z centered at a point $z_0 \in \mathbb{C}$ is an expression of the form

$$\sum_{n=0}^{\infty} c_n \cdot (z - z_0)^n = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \dots$$

We'd like to think of this as defining some complex function $f(z) = \sum_n c_n (z - z_0)^n$, but

have to worry that our series might not converge, at least for certain choices of z .

Notice the series always converges at $z = z_0$ where the value is simply z_0 .

Let's first observe that we can replace our complex series by a real series by taking absolute values. That is, let's consider

$$\sum_{n=0}^{\infty} |c_n(z-z_0)^n| = \sum_{n=0}^{\infty} |c_n| \cdot |z-z_0|^n$$

Since this is a real series, all of our convergence tests, like the ratio test, from a second semester calculus class apply.

Recall the ratio test for a real series $\sum a_n$ says that if the limit of ratios of absolute values of consecutive terms

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists and is less than 1, then the series will be absolutely convergent; if the limit is greater than 1, then the series diverges;

and if the limit equals 1, then the test is inconclusive (ie, there are instances where the series converges conditionally, instances where it converges absolutely, and instances where it diverges).

Applying this to $\sum_n |c_n(z-z_0)^n|$, we consider

$$\lim_{n \rightarrow \infty} \frac{|c_{n+1}(z-z_0)^{n+1}|}{|c_n(z-z_0)^n|} = |z-z_0| \cdot \lim_{n \rightarrow \infty} \frac{|c_{n+1}|}{|c_n|}$$

Let's suppose $\lim_{n \rightarrow \infty} \frac{|c_{n+1}|}{|c_n|}$ exists and equals some number $\frac{1}{r}$. Then we're guaranteed convergence if

$$|z-z_0|^{\frac{1}{r}} < 1$$

or, $|z-z_0| < r$. So, our series of absolute values converges if $|z-z_0| < r$. This is a disc of radius r around z_0 .

(Note if $\lim_{n \rightarrow \infty} \frac{|c_{n+1}|}{|c_n|} = 0$, then $\lim_{n \rightarrow \infty} |z-z_0| \cdot \frac{|c_{n+1}|}{|c_n|} = 0 < 1$ for all z , so we say the radius of convergence is infinite.)

Our next goal is to show if the real power series $\sum |c_n(z-z_0)^n|$ converges, then so does the complex power series $\sum c_n(z-z_0)^n$. To do this we will need the following definition and a few basic lemmas.

We say a sequence $(z_n)_{n \in \mathbb{N}}$ of complex numbers is Cauchy if for every $\varepsilon > 0$ there exists a number N so that for all $m, n > N$ we have $|z_m - z_n| < \varepsilon$. That is, a sequence is Cauchy if all entries sufficiently far into the sequence become arbitrarily close to one another.

Recall a sequence $(z_n)_{n \in \mathbb{N}}$ converges to $z_\infty \in \mathbb{C}$ if for every $\varepsilon > 0$ there is an $N > 0$ so that $|z_n - z_\infty| < \varepsilon$ for all $n > N$.

Perhaps it's not too surprising, then, that having a sequence converge and having a sequence be Cauchy are closely related.

Thm

A sequence of complex numbers, $(z_n)_{n \in \mathbb{N}}$, converges iff it is Cauchy.

If
Exercise

Using the above, we can now show that a complex series $\sum \zeta_n$, $\zeta_n \in \mathbb{C}$, will converge provided $\sum |\zeta_n|$ converges.

Thm

Suppose $\sum \zeta_n$ is a complex series, but the real series $\sum |\zeta_n|$ converges. Then $\sum \zeta_n$ converges as well.

Pf

We will show $\sum \zeta_n$ is Cauchy. Let $\varepsilon > 0$ be given. Then there exists some N so that for each $m, n > N$ we have

$$\left| \sum_{k=1}^m |\zeta_k| - \sum_{k=1}^n |\zeta_k| \right| < \varepsilon.$$

Without loss of generality, we may suppose $m > n$ and write the above inequality as

$$\left| \sum_{k=1}^m |\zeta_k| - \sum_{k=1}^n |\zeta_k| \right| = \sum_{k=n+1}^m |\zeta_k| < \varepsilon$$

Now consider

$$\left| \sum_{k=1}^m \zeta_k - \sum_{k=1}^n \zeta_k \right| = \left| \sum_{k=n+1}^m \zeta_k \right| < \sum_{k=n+1}^m |\zeta_k| < \varepsilon$$

So $\sum_{k=1}^n \zeta_k$ is a Cauchy sequence, and hence converges. $\square$

Combining this theorem with our observation about power series converging when the abs. value of their ratio of coefficients converging to $\frac{1}{r}$, we have the following.

Thm

A complex power series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ converges for all z in a disc of radius r around z_0 where $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = \frac{1}{r}$.

□

The r in the theorem above is the radius of convergence of the series, and $B_r(z_0)$ is the disc of convergence for the series. Notice the series will diverge for z with $|z - z_0| > r$. For z on the boundary of $B_r(z_0)$, the series may converge for some z , but diverge for others.

Notice that if the power series $\sum_n c_n (z - z_0)^n$ converges in $B_r(z_0)$, then in this ball it determines some function $f: B_r(z_0) \rightarrow \mathbb{C}$. It's not too hard to show this function is holomorphic by showing $\frac{\partial f}{\partial \bar{z}} = 0$.

Thm

The complex power series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ determines

a holomorphic function where it converges.

Pf

Recall (from a homework assignment) that a function f is holomorphic iff the Wirtinger derivative $\frac{\partial f}{\partial \bar{z}}$ is zero, where $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$.
Now observe

$$\begin{aligned}\frac{\partial}{\partial \bar{z}} \sum_{n=0}^{\infty} c_n (z - z_0)^n &= \sum_{n=0}^{\infty} \frac{\partial}{\partial \bar{z}} c_n (z - z_0)^n \\ &= \sum_{n=0}^{\infty} c_n \cdot \frac{1}{2} \left[\frac{\partial}{\partial x} (z - z_0)^n + i \frac{\partial}{\partial y} (z - z_0)^n \right]\end{aligned}$$

Now let's write $z = x + iy$, so

$$\begin{aligned}\frac{\partial}{\partial x} (z - z_0)^n &= \frac{\partial}{\partial x} (x + iy - z_0)^n \\ &= n(x + iy - z_0)^{n-1} \cdot \frac{\partial}{\partial x} (x + iy - z_0) \\ &= n(x + iy - z_0)^{n-1} = n(z - z_0)^{n-1}\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial y} (z - z_0)^n &= \frac{\partial}{\partial y} (x + iy - z_0)^n \\ &= n(x + iy - z_0)^{n-1} \cdot \frac{\partial}{\partial y} (x + iy - z_0) \\ &= in(x + iy - z_0)^{n-1} = in(z - z_0)^{n-1}\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial}{\partial \bar{z}} (z - z_0)^n &= \frac{1}{i} \left(\frac{\partial}{\partial x} (z - z_0)^n + i \frac{\partial}{\partial y} (z - z_0)^n \right) \\ &= \frac{1}{i} \left(n(z - z_0)^{n-1} + i (i n (z - z_0)^{n-1}) \right) \\ &= \frac{1}{i} \left(n(z - z_0)^{n-1} - n(z - z_0)^{n-1} \right) \\ &= 0\end{aligned}$$

Hence $\frac{\partial}{\partial \bar{z}} \sum_n C_n (z - z_0)^n = 0$, so the power series is holomorphic. $\square$

So, power series in z always determine holomorphic functions, where the series converges. We can go further, though, and show that every holomorphic function can be locally represented by a power series. To do this, it will be helpful to recall the definition of Taylor polynomials from a second semester calculus class, but adapted for complex variables.

Given a holomorphic function $f: U \rightarrow \mathbb{C}$ and a point $z_0 \in U$, we define the n^{th} -order Taylor polynomial of f centered at z_0 to be the unique degree n polynomial $p(z)$ so that

$$\begin{aligned}
 p(z_0) &= f(z_0) \\
 p'(z_0) &= f'(z_0) \\
 p''(z_0) &= f''(z_0) \\
 &\vdots \\
 p^{(n)}(z_0) &= f^{(n)}(z_0)
 \end{aligned}$$

Thinking of $p(z)$ as having unknown coefficients $p(z) = C_n z^n + C_{n-1} z^{n-1} + \dots + C_1 z + C_0$, this gives us a system of $n+1$ linear equations in $n+1$ unknowns, so we expect this to have a unique solution. The solution is easiest to discern if we rewrite our polynomial as

$$p(z) = C_n (z - z_0)^n + C_{n-1} (z - z_0)^{n-1} + \dots + C_1 (z - z_0) + C_0.$$

Then we can compute

$$p(z_0) = C_0$$

$$\begin{aligned}
 p'(z) &= n C_n (z - z_0)^{n-1} + (n-1) C_{n-1} (z - z_0)^{n-2} + \dots + 2 C_1 (z - z_0) + C_1 \\
 \Rightarrow p'(z_0) &= C_1
 \end{aligned}$$

$$\begin{aligned}
 p''(z) &= n(n-1) C_n (z - z_0)^{n-2} + (n-1)(n-2) C_{n-1} (z - z_0)^{n-3} + \dots + 3 \cdot 2 \cdot C_3 (z - z_0) + 2 C_2 \\
 &\quad + 2 C_2
 \end{aligned}$$

$$\Rightarrow p''(z_0) = 2 C_2$$

In general, for $k \leq n$, $p^{(k)}(z_0) = k! \cdot C_k$
Thus our system of equations above becomes

$$\begin{aligned}C_0 &= f(z_0) \\C_1 &= f'(z_0) \\2C_2 &= f''(z_0) \\&\vdots \\k! \cdot C_k &= f^{(k)}(z_0) \\&\vdots \\n! \cdot C_n &= f^{(n)}(z_0)\end{aligned}$$

Which we easily see means $C_k = \frac{f^{(k)}(z_0)}{k!}$,

so the n^{th} -order Taylor polynomial for our holomorphic $f: U \rightarrow \mathbb{C}$ in the function $p: \mathbb{C} \rightarrow \mathbb{C}$ given by

$$p(z) = \sum_{k=0}^n \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$$

We'd like to go from the Taylor polynomial to the Taylor series,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$$

But now we have to worry about convergence.

Math 424, Spring 2024
Lecture 14 - Power Series (Continued)

In the last lecture we observed that the complex power series

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n$$

will converge in $B_r(z_0)$ if $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = 1/r$.

Let's use this to help elucidate some familiar power series of real numbers from calculus.

First recall the familiar geometric series: given a real number x , we consider the infinite sum

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

Suppose momentarily this converges to the value S . We can then think of S as the limit of partial sums $S = \lim_{N \rightarrow \infty} S_N$ where

$$S_N = \sum_{n=0}^N x^n = 1 + x + x^2 + \dots + x^{N-1} + x^N$$

Observe that $S_N - xS_N$ telescopes, leaving the first and last terms,

$$S_N - xS_N = (1 + x + x^2 + \dots + x^N + x^{N+1}) - (x + x^2 + \dots + x^N + x^{N+1})$$

$$= 1 - x^{N+1}$$

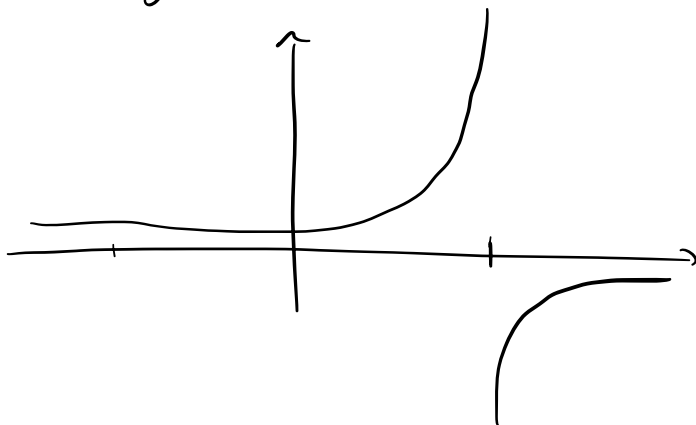
$$\Rightarrow S_N(1-x) = 1 - x^{N+1}$$

$$\Rightarrow S_N = \frac{1 - x^{N+1}}{1 - x}$$

Taking the limit as N goes to infinity we see this value is finite iff $|x| < 1$, in which case we have

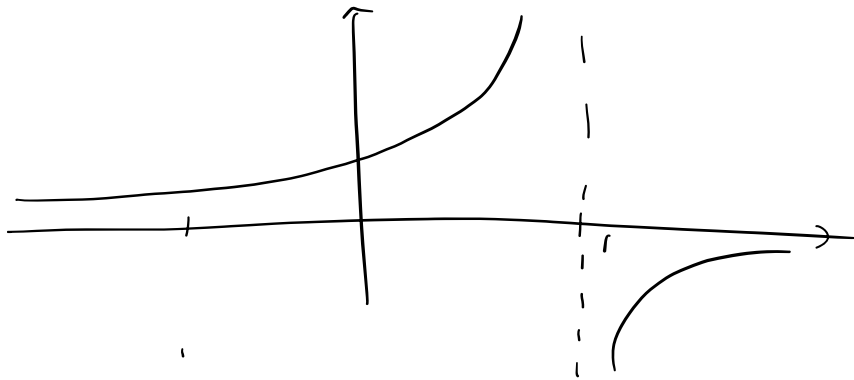
$$S = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if } |x| < 1.$$

Thinking of this as a power series, $\sum_{n=0}^{\infty} \underbrace{1}_{c_n} \cdot \underbrace{(x-0)^n}_{x^n}$, we see the function $\frac{1}{1-x}$ is represented as a convergent power series in an open interval of radius 1 around 0. Notice as x approaches 1 from the left, or -1 from the right, $\frac{1}{1-x}$ goes to infinity.



This divergence to infinity seems to account for

the radius of convergence of $\frac{1}{1-x}$ around 0, which we can easily double-check using the ratio test as $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{1} \right| = 1$. The function $\frac{1}{1-x}$ is defined outside of $(-1, 1)$ however — the function is well-defined on $\mathbb{R} \setminus \{-1, 1\}$



We can also try to represent the function as a power series centered at some other point. For example, we can compute the Taylor series of $f(x) = \frac{1}{1-x}$ centered at $x_0 = -6$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(-6)}{n!} (x - (-6))^n$$

Here

$$\begin{aligned} f(x) &= \frac{1}{1-x} = (1-x)^{-1} \\ f'(x) &= -(1-x)^{-2} \cdot (-1) = (1-x)^{-2} \\ f''(x) &= -2(1-x)^{-3} \cdot (-1) = 2(1-x)^{-3} \\ f'''(x) &= 2 \cdot (-3) \cdot (1-x)^{-4} \cdot (-1) = 6(1-x)^{-4} \\ &\vdots \\ f^{(n)}(x) &= n! (1-x)^{-(n+1)} \end{aligned}$$

Thus our series is

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{f^{(n)}(-6)}{n!} (x-(-6))^n &= \sum_{n=0}^{\infty} \frac{n! (1-(-6))^{-(n+1)}}{n!} (x+6)^n \\ &= \sum_{n=0}^{\infty} 7^{-(n+1)} \cdot (x+6)^n\end{aligned}$$

The radius of convergence can be computed using the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{7^{-(n+1)}}{7^{-n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{7^n}{7^{n+1}} \right| = \frac{1}{7}$$

So the radius of convergence is 7; the series converges if $|x+6| < 7$, so $-13 < x < 1$. Again, the convergence is "blocked" by this asymptote at $x=1$.

Let's modify the example, slightly. Notice by replacing x in $\frac{1}{1-x} = \sum x^n$ with $-x^2$, we get

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

This should converge when $|(-x^2)| < 1$, so $|x^2| < 1$, so again $|x| < 1$, as confirmed by the ratio test:

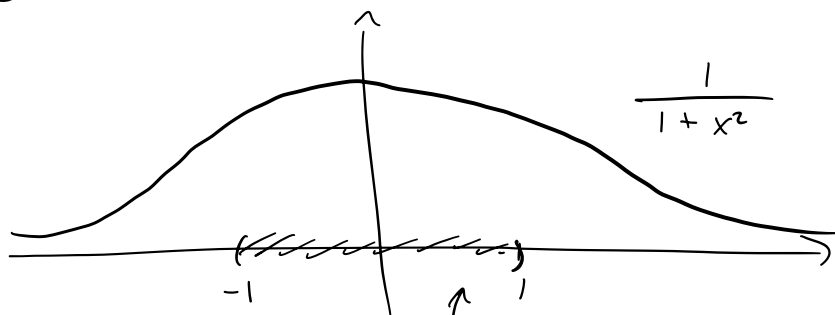
$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{(-1)^n x^{2n}} \right| < 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{|x|^{2n+2}}{|x|^{2n}} < 1$$

$$\Rightarrow |x^2| < 1.$$

and $|x^2| < 1$ iff $|x| < 1$.

Notice, however, that $\frac{1}{1+x^2}$ is defined at every real x .



Interval of convergence
for $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ is $(-1, 1)$.

We can even try to find the series expansion of $\frac{1}{1+x^2}$ at a point outside of $(-1, 1)$, like $x_0 = 1$. The computation is tedious, but we can show the radius of convergence for the Taylor series expansion of $\frac{1}{1+x^2}$ centered at $x=1$ is $\sqrt{2}$.

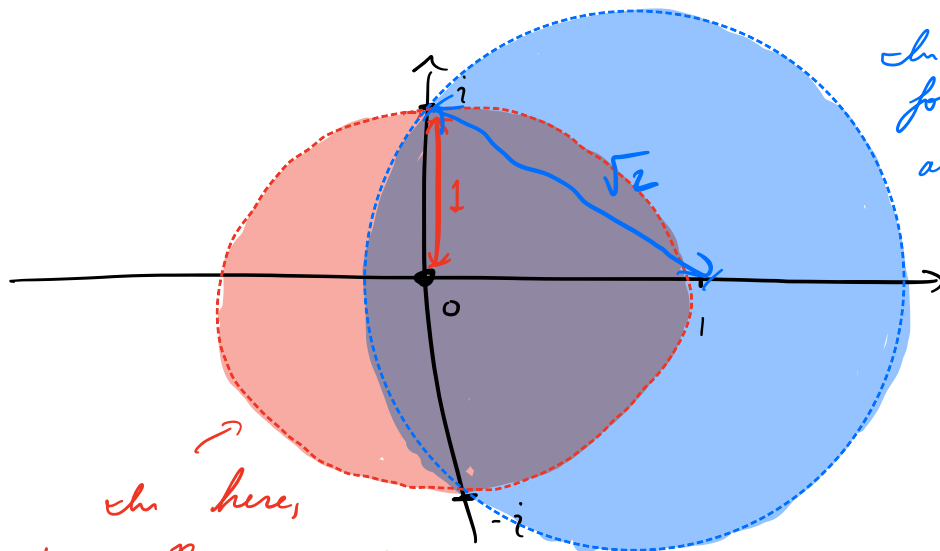
Why does our perfectly nice real valued

function $\frac{1}{1+x^2}$ which is defined for all real numbers have these finite radii of convergence? What's the significance of a number like $\sqrt{2}$?

The answers to these questions are dramatically cleared up if we "upgrade" from a real x to a complex z . That is consider the function $f(z) = \frac{1}{1+z^2}$. We can show that the complex Taylor series for this function at $z=0$ and $z=i$ have the same coefficients as our real series expansion of $\frac{1}{1+x^2}$ and so have the same radius of convergence. However, let's observe this complex version of the function does have some problematic points:

$$\begin{aligned}1 + z^2 &= 0 \\ \Rightarrow z^2 &= -1 \\ \Rightarrow z &= \pm\sqrt{-1} = \pm i\end{aligned}$$

The radii of convergence above are exactly the distance from our centers, $z=0$ and $z=i$ to the nearest of these problematic points



the here, series
for $\frac{1}{1+z^2}$ centered
at $z=1$ converges

the here,

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

Note $\frac{1}{1+z^2}$ is holomorphic in $\mathbb{C} \setminus \{\pm i\}$. This hints at the following remarkable fact: if f is holomorphic in U , then for every ball $B_r(z_0) \subseteq U$, f has a convergent power series representation inside $B_r(z_0)$. This kind of result is very special about complex functions and is not true for real functions. E.g., it can be shown that the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

is infinitely-differentiable (a prerequisite to having a power series representation), but can not be represented as a power series around $x=0$.

The easiest way to prove this result (that holomorphic functions are locally given by power series) is to use a result called the "Cauchy integral formula" which we learn about soon. Once we have the Cauchy integral formula, we'll come back to show this remarkable result.

Math 424, Spring 2024
Lecture 15 - Power series, pt. 3

In this lecture we will momentarily finish up our discussion of power series. We'll mention power series again later, after we've developed some more machinery which will make certain results easier to prove, but for now we'll tie up a few loose ends.

First, let's observe that power series representations of functions are unique. That is suppose we had two ostensibly different power series centered at z_0 which represented the same function $f(z)$ in the overlapping discs of convergence. E.g., suppose

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n = f(z) = \sum_{n=0}^{\infty} c'_n (z - z_0)^n$$

for all z in some disc $B_r(z_0)$. Then we can show $c_n = c'_n$ for each n . The idea is to observe first that $c_0 = c'_0$ as

$$f(z_0) = \sum_{n=0}^{\infty} c_n (z_0 - z_0)^n = c_0 + c_1 \cdot 0 + c_2 \cdot 0^2 + \dots = c_0$$

and similarly, $f(z_0) = \sum_{n=0}^{\infty} c'_n (z - z_0)^n = c'_0$.

Now notice

$$\begin{aligned} f'(z) &= \sum_{n=1}^{\infty} n c_n (z-z_0)^{n-1} = c_1 + 2c_2(z-z_0) + 3c_3(z-z_0)^2 + \dots \\ &= \sum_{n=1}^{\infty} n c'_n (z-z_0)^{n-1} = c'_1 + 2c'_2(z-z_0) + 3c'_3(z-z_0)^2 + \dots \end{aligned}$$

Plugging in z_0 we see $c_1 = f'(z_0) = c'_1$.

Taking the second derivative,

$$\begin{aligned} f''(z) &= \sum_{n=2}^{\infty} n(n-1) c_n (z-z_0)^{n-2} = 2c_2 + 6c_3(z-z_0) + 12c_4(z-z_0)^2 + \dots \\ &= \sum_{n=2}^{\infty} n(n-1) c'_n (z-z_0)^{n-2} = 2c'_2 + 6c'_3(z-z_0) + 12c'_4(z-z_0)^2 + \dots \end{aligned}$$

Note $f''(z_0) = 2c_2 = 2c'_2 \Rightarrow c_2 = c'_2 = \frac{f''(z_0)}{2}$

Continuing to take derivatives and evaluating $f^{(n)}(z_0)$ we will see

$$f^{(n)}(z_0) = n! c_n = n! c'_n$$

$$\Rightarrow c_n = c'_n = \frac{f^{(n)}(z_0)}{n!}$$

Observe this tells us power series in general are necessarily the Taylor series of the functions they represent!

We can use this to prove the following important theorem:

Thm (Identity Theorem for Power Series)

Suppose $f(z)$ and $g(z)$ are two functions represented by complex power series centered at z_0 ,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

$$g(z) = \sum_{n=0}^{\infty} d_n (z - z_0)^n$$

If there exists a sequence $(z_n)_{n \in \mathbb{N}}$ contained in the intersection of the domains of f and g with $\lim_{n \rightarrow \infty} z_n = z_0$ and $f(z_n) = g(z_n)$ for all n , then $f(z) = g(z)$. □

We will actually prove this as a corollary of the following

Thm

If $f(z)$ is a function represented by a complex power series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ and $f(z_n) = 0$ for each n in some sequence approaching z_0 , then each $c_n = 0$ and $f(z)$ is the constant zero function.

Pf

Notice by continuity, $f(z_0) = \lim_{n \rightarrow \infty} f(z_n) = 0$

and so $c_0 = 0$. Thus $f(z)$ may be expressed

as

$$f(z) = \sum_{n=1}^{\infty} c_n (z - z_0)^n \quad (\text{as } c_0 = 0)$$

$$\Rightarrow (z - z_0) f(z) = \sum_{n=1}^{\infty} c_n (z - z_0)^n$$

$$= \sum_{n=0}^{\infty} c_{n+1} (z - z_0)^n$$

$$= c_1 + c_2 (z - z_0) + c_3 (z - z_0)^2 + \dots$$

Call this new function $f_1(z)$. Note $f_1(z)$ is continuous and $f_1(z_n) = 0$ for each z_n as well since $f_1(z_n) = (z_n - z_0) \cdot f(z_n) = 0$. Thus

repeating the same continuity argument above, $f_1(z_0) = 0$

so $c_1 = 0$. Repeating this process indefinitely (which we can make precise using induction), we see each $c_n = 0$.

□

Pf of the identity theorem

Suppose $f(z_n) = g(z_n)$ for all z_n along some sequence approaching z_0 . Consider the function

$$h(z) := f(z) - g(z) = \sum_{n=0}^{\infty} (c_n - d_n) (z - z_0)^n$$

Note $h(z_n) = 0$ for all z_n in the sequence, so by the theorem above h is the constant zero function and each coefficient $c_n - d_n$ in its power series representation is zero, so $c_n = d_n$ and $f = g$. $\square$

Notice this theorem implies that if a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is represented by a real power series, then the function can be "upgraded" to a complex function $f: \mathbb{C} \rightarrow \mathbb{C}$, given by a power series in z with the same coefficients as the original real power series in exactly one way.

For example, the function $x \mapsto e^x$ is defined on $\mathbb{R}$ by

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

We define the complex exponential e^z to thus be

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

This is the only function represented by a power series on $\mathbb{C}$ which agrees with e^x for $x \in \mathbb{R}$!

We can verify that familiar properties of the real exponential carry over to the complex exponential, such as

$$e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} (z_1 + z_2)^n &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^n \binom{n}{k} z_1^k z_2^{n-k} \right) \\ &= 1 + (z_1 + z_2) + \left(\frac{z_1^2 + 2z_1 z_2 + z_2^2}{2!} \right) + \left(\frac{z_1^3 + 3z_1^2 z_2 + 3z_1 z_2^2 + z_2^3}{3!} \right) + \dots \\ &= \left(1 + z_1 + \frac{z_1^2}{2!} + \frac{z_1^3}{3!} + \dots \right) \left(1 + z_2 + \frac{z_2^2}{2!} + \frac{z_2^3}{3!} + \dots \right) \\ &= e^{z_1} \cdot e^{z_2} \end{aligned}$$

This can be used to compute complex exponentials by writing $z = x + iy$, so

$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos(y) + i \sin(y))$$

E.g., $e^{3+4i} = e^3 [\cos(4) + i \sin(4)]$

Notice if the imaginary part increases by $2\pi n$, $n \in \mathbb{Z}$, the e^z is unchanged:

$$\begin{aligned} e^{3+i(4+2\pi n)} &= e^3 (\cos(4+2\pi n) + i \sin(4+2\pi n)) = e^3 [\cos(4) + i \sin(4)] \\ &= e^{3+4i} \end{aligned}$$

in general,

$$e^{z+2\pi n} = e^z e^{i2\pi n} = e^z \cdot (\cos(2\pi n) + i\sin(2\pi n)) = e^z \cdot 1 = e^z$$

Similarly, $\sin(x)$ and $\cos(x)$ have ^(real) power series representation

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

We use this to define complex sine and cosine

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad \cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

Note that the radius of convergence of our power series for e^z , $\sin(z)$, and $\cos(z)$ are all infinite, and so these are well-defined holomorphic functions defined on all of $\mathbb{C}$. We can use these series representations to justify several identities (homework):

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

$$e^{iz} = \cos(z) + i\sin(z) \quad \sin^2(z) + \cos^2(z) = 1$$

Having discussed the complex exponential, the next natural function to consider is the complex logarithm.

For real x , we define $\log(x)$ as the inverse of e^x . We can not define a complex $\log(z)$ as "the" inverse of e^z since e^z is not one-to-one—namely, $e^z = e^{z + i2\pi}$.

We could reasonably use power series to define the complex log. Recall the Maclaurin series of $\log(1+x)$, $x \in \mathbb{R}$, is

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

This converges for $x \in (-1, 1)$. We can then define

$$\log(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$$

which will converge for $z \in B_1(0)$.

$$\begin{aligned} \text{E.g., } \log\left(\frac{3}{2} + i\frac{1}{2}\right) &= \log\left(1 + \frac{1}{2} + \frac{i}{2}\right) \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{1}{2} + \frac{i}{2}\right)^n}{n} \\ &\approx 0.45815 + 0.32175i \end{aligned}$$

This power series only allows us to compute $\log(z)$ for $z \in B_1(1)$ (this is $\log(1+z)$ for $z \in B_1(0)$). We'd like to extend our definition to more of the complex plane, but the fact we can't define $\log(z)$ at zero creates a problem.

(As $z \rightarrow -1$, our series expansion for $\log(1+z)$ becomes

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{-1} \cdot \frac{1}{n}$$
$$= - \sum_{n=1}^{\infty} \frac{1}{n}$$

which goes to $-\infty$.)

Math 424, Spring 2024

Lecture 16 - The complex logarithm and analytic continuation

In this lecture we'll try to make sense of a notion of a complex logarithm. We'd like to do this in some way that agrees with our notion of the logarithm for real numbers, and so a natural starting point might be to "upgrade" our real logarithm to the complex plane by mimicking the definition of the real log.

For real numbers, the function $\ln(x)$ is defined as the inverse of e^x . Unfortunately this definition doesn't quite work for a complex log since the complex exponential e^z is not injective: as we'd seen in the last lecture, e^z is periodic w/period $2\pi i$ — $e^{z+2\pi i} = e^z$.

In general, we say a function $f(z)$ such that $e^{f(z)} = z$ is a branch of the complex logarithm. Observe there is no choice of z so that $e^z = 0$:

$$\begin{aligned} e^z &= e^{x+iy} = e^x (\cos(y) + i\sin(y)) = 0 \\ &\Rightarrow \cos(y) + i\sin(y) = 0 \\ &\Rightarrow \cos(y) = \sin(y) = 0 \end{aligned}$$

But $|\cos(y) + i\sin(y)| = \cos^2(y) + \sin^2(y) = 1$ for all y .

So, any branch of the complex log we wish to define will necessarily be undefined at 0.

Our goal now should be to find "a" branch of the complex log instead of "the" complex log.

Using our knowledge of power series, one attractive approach might be to use a power series representation of $\log(x)$, but then use a complex variable to get a holomorphic function.

It is an easy exercise in calculus to show that the Taylor series for $\log(x)$ centered at $x=1$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot (x-1)^n$$

which has radius of convergence 1. Let's define a function $L(z)$ (not standard notation) to define a function

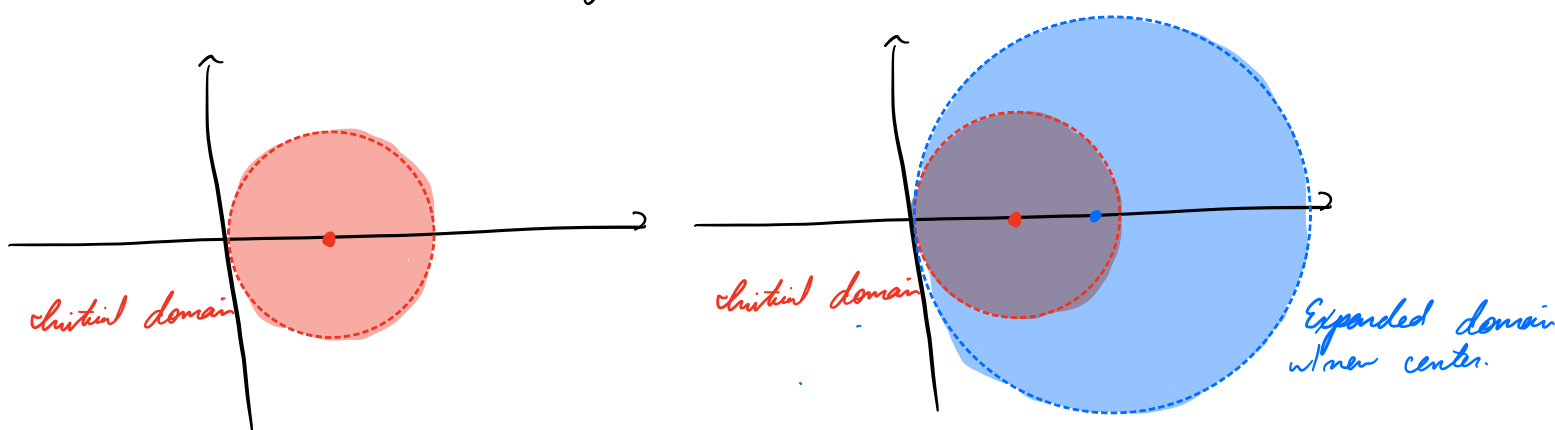
$$L(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n$$

From the last few lectures, we can see this defines a holomorphic function in the disc of

convergence of this series, which is $B_1(1)$. Furthermore, for real x in this ball (i.e., $0 < x < 2$), the value of $L(x)$ will agree with $\log(x)$.

Of course, the "problem" with this $L(x)$ is that it is only defined in $B_1(1)$. We'd like to extend $L(z)$ to be defined in as much of the complex plane as possible. One approach to extending the domain of $L(z)$ is to use a technique called "analytic continuation."

The idea is to take our series centered at 1 and use it to compute the series expansion around some other point within the initial series' disc of convergence. If the "new" center for the series is further from any singularities of our function, then the radius of convergence will be larger, so we can define the function on a larger domain.



In order to compute the new series centered at

another point, we need to be able to compute the derivatives of the function. Since we can differentiate a power series by differentiating each term, in principle we can determine our derivatives just using the power series. However, it will be helpful to note the following results as it will make our computation for expanding the domain of $L(z)$ a bit easier.

Thm

Suppose U and V are open subsets of $\mathbb{C}$ and $g: U \rightarrow V$, $f: V \rightarrow U$ are two holomorphic functions satisfying $g(f(z)) = z$ for all $z \in V$. Then

$$f'(z) = \frac{1}{g'(f(z))}$$

Pf

$$g(f(z)) = z$$

$$\Rightarrow \frac{d}{dz} g(f(z)) = \frac{d}{dz} z$$

$$\Rightarrow g'(f(z)) \cdot f'(z) = 1$$

$$\Rightarrow f'(z) = \frac{1}{g'(f(z))}$$

□

As a corollary, if $f(z)$ is any branch of the complex exponential ($e^{f(z)} = z$), then $f'(z) = \frac{1}{z}$.

Continuing to differentiate, a nice pattern emerges

$$f'(z) = \frac{1}{z} = z^{-1}$$

$$f''(z) = -z^{-2}$$

$$f'''(z) = 2z^{-3}$$

$$f^{(4)}(z) = -6z^{-4}$$

⋮

$$f^{(n)}(z) = (-1)^{n+1} (n-1)! \cdot z^{-n}$$

Thus if we take our series expansion for $L(z)$ centered at $\xi_0 = 1$ and compute the series expansion about some new ξ_1 in the disc of convergence of $L(z)$, we will obtain

$$\begin{aligned} L(z) &= \sum_{n=0}^{\infty} \frac{L^{(n)}(\xi_1)}{n!} (z - \xi_1)^n \\ &= L(\xi_1) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot (n-1)! \cdot \xi_1^{-n}}{n!} (z - \xi_1)^n \\ &= L(\xi_1) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot \xi_1^n} (z - \xi_1)^n \end{aligned}$$

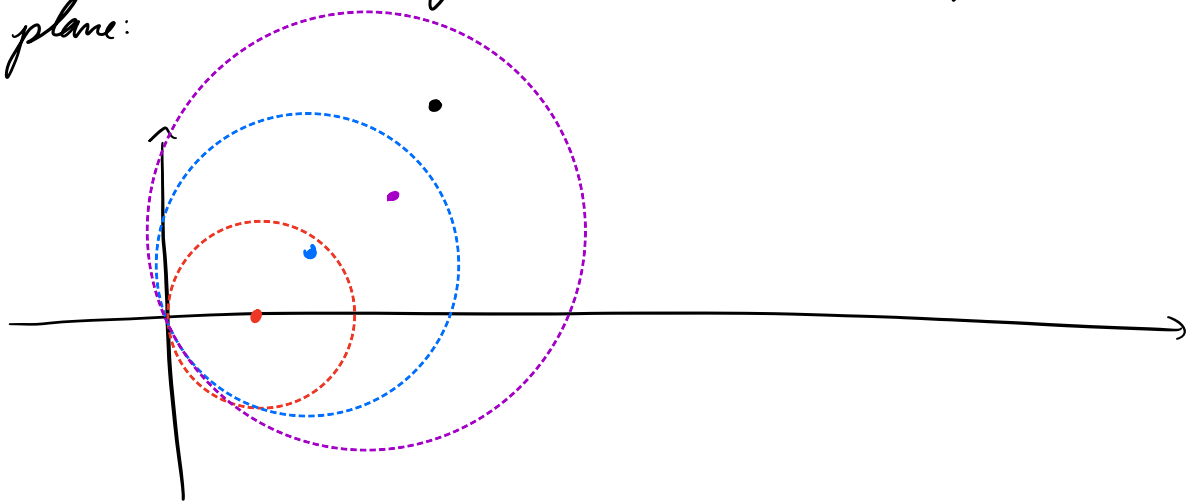
This series converges in $B_{|\xi_1|}(\xi_1)$, as we can verify w/ the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+2}}{(n+1) \cdot \xi_1^{n+1}} (z - \xi_1)^{n+1}}{\frac{(-1)^{n+1}}{n \cdot \xi_1^n} (z - \xi_1)^n} \right| < 1$$

$$\Rightarrow |z - \xi_1| \cdot \lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot \frac{1}{|\xi_1|} < 1$$

$$\Rightarrow |z - \xi_1| < |\xi_1|$$

It would seem like this gives us a reasonable way to define a complex logarithm on $\mathbb{C} \setminus \{0\}$: start from the series expansion agreeing w/ $\log(x)$ on $(0, 2)$, and then perform analytic continuation repeatedly to expand the domain of $L(z)$ to other points in the plane:



This does give us a reasonable notion of a complex log, unless we try to extend the domain so much that it encircles 0.

As we can verify by doing the tedious computation

by hand (or on a computer), if we try to extend our definition of $L(z)$ by going in a circle around 0 to come back to 1, we will compute two different values of $L(1)$ — the initial series tells us $L(1) = 0$, but the "new" series we develop after going in a loop around 0 will tell us $L(1) = 2\pi i$.

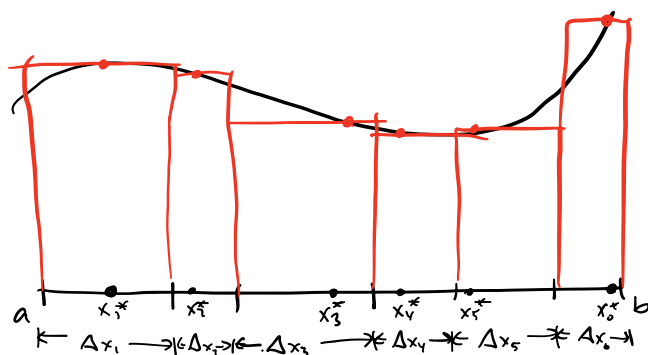
(See supplemental Sage code)

Math 424, Spring 2024
Lecture 17 - Complex Integration

Recall that the (Riemann) integral of a function $f(x)$ defined on an interval $[a, b] \subseteq \mathbb{R}$ is defined to be the limit of Riemann sums,

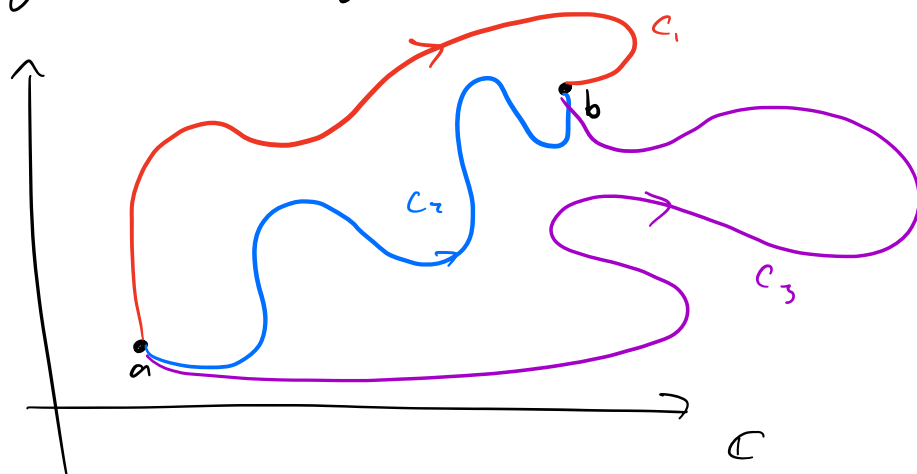
$$\sum_{i=1}^n f(x_i^*) \Delta x_i$$

Here, we imagine the interval $[a, b]$ as being divided into n subintervals, and we associate a number, $f(x_i^*) \Delta x_i$, to each segment, where Δx_i is the length of the i^{th} interval, and x_i^* is some point in that interval.



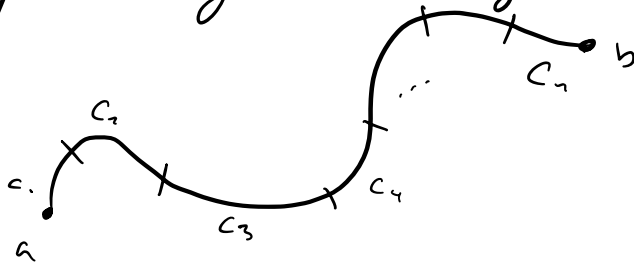
We obtain $\int_a^b f(x) dx$ by taking the limit of these Riemann sums as the size of the intervals shrinks to zero (which implies the number of intervals grows to infinity).

We can try to mimic this procedure for complex functions $f(z)$, but note that there can be several choices of how to get from point a to point b . (In $\mathbb{R}$, there's essentially only one way to get from a to b .)

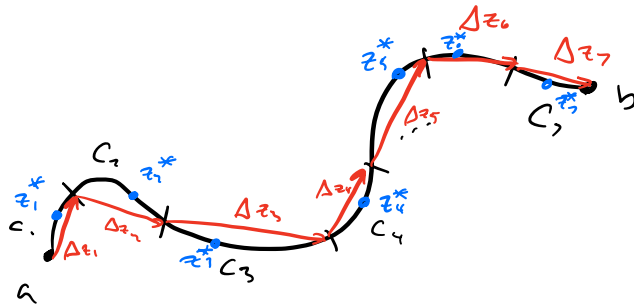


A priori, the choice of a curve from a to b can alter the value we compute, and so for a general complex function $f(z)$, our notion of an integral will depend not only on the points a and b , but also how we get from a to b .

Once a choice of curve C from a to b is chosen, we can define a Riemann sum of $f(z)$ along C as follows: We segment C into n pieces, say C_1 through C_n .



For each piece C_k , let z_k^* be any choice of point on C_k , and let Δz_k be the vector (thought of as a complex number) joining the endpoints of C_k



We then consider the sum

$$\sum_{k=1}^n f(z_k^*) \Delta z_k$$

Note if we write $z_k^* = x_k^* + iy_k^*$, $f(x+iy) = u(x,y) + iv(x,y)$, and $\Delta z_k = \Delta x_k + i\Delta y_k$, the quantity above can be expressed as

$$\begin{aligned} & \sum_{k=1}^n [u(x_k^*, y_k^*) + iv(x_k^*, y_k^*)] \cdot [\Delta x_k + i\Delta y_k] \\ &= \sum_{k=1}^n \left(u(x_k^*, y_k^*) \Delta x_k + iu(x_k^*, y_k^*) \Delta y_k + iv(x_k^*, y_k^*) \Delta x_k - v(x_k^*, y_k^*) \Delta y_k \right) \\ &= \sum_{k=1}^n \left[(u(x_k^*, y_k^*) \Delta x_k - v(x_k^*, y_k^*) \Delta y_k) + i(v(x_k^*, y_k^*) \Delta x_k + u(x_k^*, y_k^*) \Delta y_k) \right] \end{aligned}$$

Taking the limit as the length of each segment C_k

goes to zero, we have the complex integral,

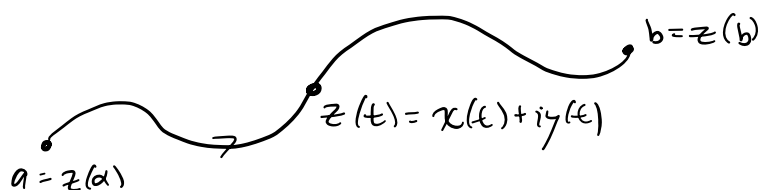
$$\int_C f(z) dz$$

which we can express in terms of the real and imaginary parts of f as

$$\int_C u dx - v dy + i \int_C v dx + u dy$$

Of course, these expressions aren't terribly useful for actually evaluating $\int_C f(z) dz$. To get something we have some hope of actually computing, it's helpful to use a parametrization of C .

So suppose C is parametrized by $z(t) = x(t) + iy(t)$ with $\alpha \leq t \leq \beta$



Our earlier Riemann sums can be written as

$$\sum_k \left[u(x(t_k^*), y(t_k^*)) (x(t_k) - x(t_{k-1})) - v(x(t_k^*), y(t_k^*)) (y(t_k) - y(t_{k-1})) \right. \\ \left. + i \left(v(x(t_k^*), y(t_k^*)) (x(t_k) - x(t_{k-1})) + u(x(t_k^*), y(t_k^*)) (y(t_k) - y(t_{k-1})) \right) \right]$$

Assuming $x, y: [a, b] \rightarrow \mathbb{R}$ are differentiable, then by the mean value theorem there is some t'_k so that

$$x'(t'_k) = \frac{x(t_k) - x(t_{k-1})}{t_k - t_{k-1}}$$

$$\begin{aligned} \Rightarrow x(t_k) - x(t_{k-1}) &= x'(t'_k)(t_k - t_{k-1}) \\ &= x'(t'_k) \Delta t_k \end{aligned}$$

and the sum becomes

$$\begin{aligned} &\sum_k [u(x(t_k), y(t_k)) x'(t'_k) - v(x(t_k), y(t_k)) y'(t'_k)] \Delta t_k \\ &+ i \sum_k [v(x(t_k), y(t_k)) x'(t'_k) + u(x(t_k), y(t_k)) y'(t'_k)] \Delta t_k \end{aligned}$$

which in the limit becomes

$$\begin{aligned} &\int_a^b [u(x(t), y(t)) x'(t) - v(x(t), y(t)) y'(t)] dt \\ &+ i \int_a^b [v(x(t), y(t)) x'(t) + u(x(t), y(t)) y'(t)] dt \end{aligned}$$

Which corresponds to the following multivariable line integral:

$$\int_C u(x, y) dx - v(x, y) dy + i \int_C v(x, y) dx + u(x, y) dy$$

To quickly recap, given a complex function $f: U \rightarrow \mathbb{C}$ given by $f(x+iy) = u(x,y) + iv(x,y)$, the integral of f over a curve C contained in U is defined as a limit of Riemann sums, analogous to the definition for a real function integrated over an interval,

$$\int_C f(z) dz = \lim_{|P| \rightarrow 0} \sum_{k=1}^{n_p} f(z_k^*) \Delta z_k$$

which can also be expressed as a sum of multivariable line integrals,

$$\int_C f(z) dz = \int_C u dx - v dy + i \int_C v dx + u dy$$

These integrals become "regular" calc. 1 integrals once a parametrization $z(t) = x(t) + iy(t)$, $\alpha \leq t \leq \beta$, is given:

$$\int_C f(z) dz = \int_{\alpha}^{\beta} f(z(t)) \cdot z'(t) dt$$

$$\int_C u dx - v dy + i \int_C v dx + u dy$$

$$= \int_{\alpha}^{\beta} [u(x(t), y(t)) x'(t) - v(x(t), y(t)) y'(t)] dt + i \int_{\alpha}^{\beta} [v(x(t), y(t)) x'(t) + u(x(t), y(t)) y'(t)] dt$$

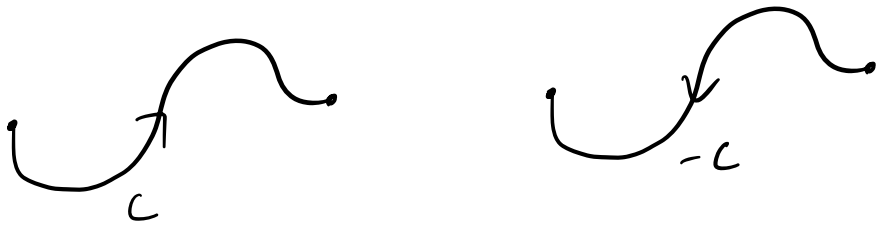
For example, if C is the section of the parabola $y=x^2$ from $(0,0)$ to $(2,4)$ in the plane parametrized by $z(t) = x(t) + iy(t) = t + it^2$, $0 \leq t \leq 2$, then we can compute $\int_C 3z^2 dz$ as

$$\begin{aligned}
 \int_C 3z^2 dz &= \int_0^2 3 \underbrace{(t + it^2)^2}_z \cdot \underbrace{(1 + i2t)}_{dz = dx + i dy} dt \\
 &= \int_0^2 3(t^2 + i2t^3 - t^4) \cdot (1 + i2t) dt \\
 &= 3 \int_0^2 (t^2 + i2t^3 - t^4 + i2t^3 - 4t^4 - i2t^5) dt \\
 &= 3 \int_0^2 [(t^2 - 5t^4) + i(4t^3 - 2t^5)] dt \\
 &= 3 \left[\frac{t^3}{3} - t^5 + i(t^4 - \frac{1}{3}t^6) \right] \Big|_0^2 \\
 &= 3 \left[\frac{8}{3} - 32 + i(16 - \frac{64}{3}) \right] \\
 &= (8 - 96) + i(48 - 64) \\
 &= -88 - 16i
 \end{aligned}$$

Remark

Observe z^3 evaluated at the endpoints of our curve, 0 and $2+i4$, are 0 and $-88-16i$.

Recall that curves in the plane are given an "orientation", a choice of forwards/backwards along the curve. If C is an oriented curve, $-C$ refers to the same set of points but with the opposite orientation:



Changing the orientation of a curve will negate the value of any integral over that curve.

Prop

For any oriented curve C and any complex function f ,

$$\int_{-C} f(z) dz = - \int_C f(z) dz$$

Pf

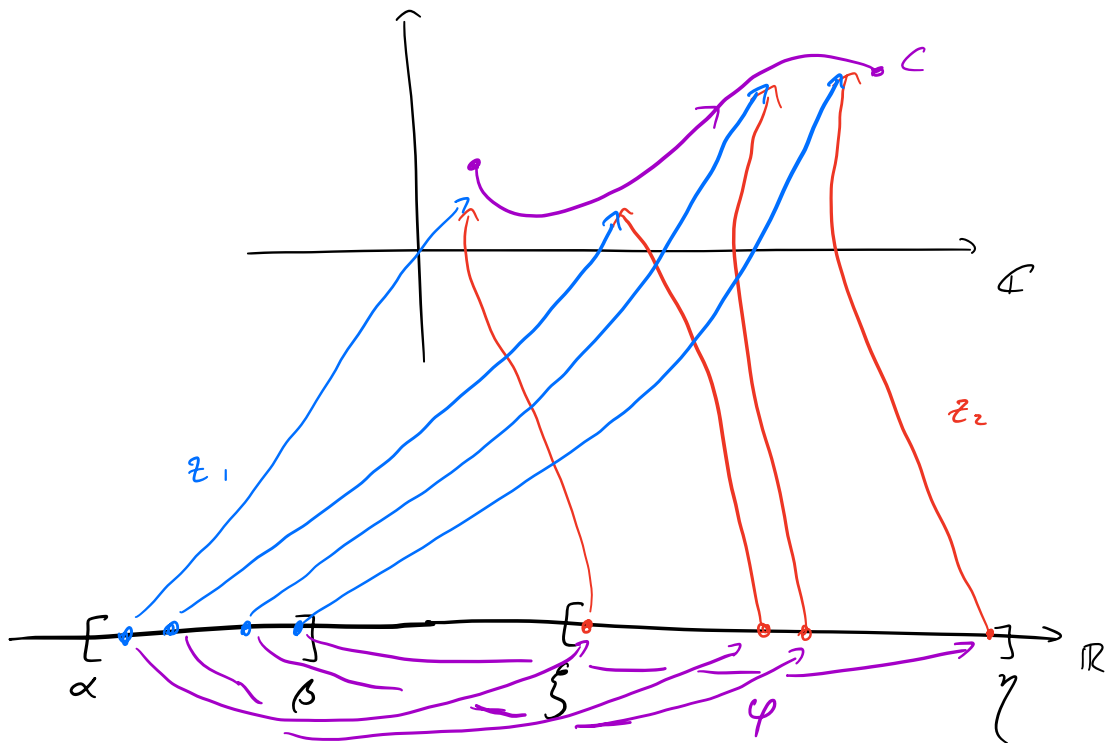
Exercise

The integrals $\int_C f(z) dz$ we have defined are sometimes called contour integrals or path integrals. As we will see, these integrals have some amazing (and surprising) properties, but let's first

address one conceivable issue about computing these integrals using parametrization: does the value we calculate depend on our choice of parametrization? That is, if $z_1(t) = x_1(t) + iy_1(t)$, $\alpha \leq t \leq \beta$, and $z_2(t) = x_2(t) + iy_2(t)$, $\xi \leq t \leq \eta$, are two parametrizations of an oriented curve C , is it true that

$$\int_{\alpha}^{\beta} f(z_1(t)) z_1'(t) dt \text{ equals } \int_{\xi}^{\eta} f(z_2(t)) z_2'(t) dt$$

To see these are equal, let's note there exists a function $\varphi(t)$ from $[\alpha, \beta]$ to $[\xi, \eta]$ so that $z_1(t) = z_2(\varphi(t))$



To justify the existence of such a φ , let's suppose z_1 and z_2 are injective, so they have a well-defined inverse function on their common

image, C . Then we can simply define $\varphi = z_2^{-1} \circ z_1$, so $z_2 \circ \varphi = z_2 \circ z_2^{-1} \circ z_1 = z_1$.

Now consider

$$\begin{aligned} & \int_{\alpha}^{\beta} f(z_1(t)) \cdot z_1'(t) dt \\ &= \int_{\alpha}^{\beta} f(z_2(\varphi(t))) \cdot \frac{d}{dt} z_2(\varphi(t)) dt \\ &= \int_{\alpha}^{\beta} f(z_2(\varphi(t))) \cdot z_2'(\varphi(t)) \cdot \varphi'(t) dt \end{aligned}$$

Now perform the substitution $u = \varphi(t)$, $du = \varphi'(t) dt$ and our integral becomes

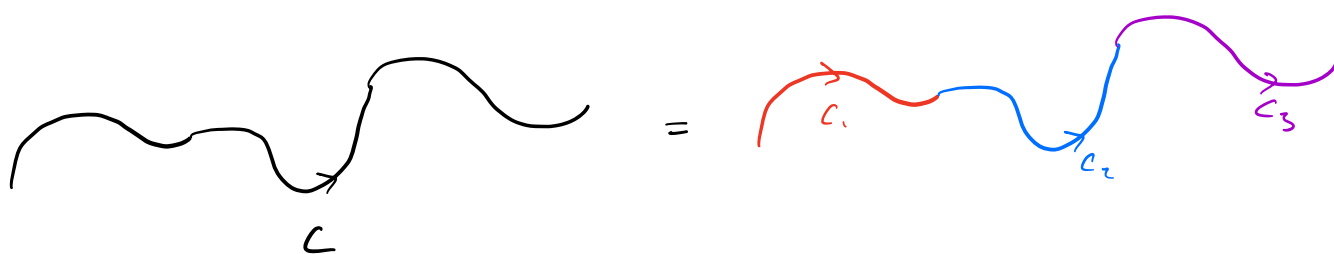
$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(z_2(u)) z_2'(u) du = \int_{\xi}^{\eta} f(z_2(t)) z_2'(t) dt$$

Thus we are justified in using any parametrization we'd like when computing a contour integral.

As a consequence, notice we can integrate over a collection of curves concatenated together by summing the integrals over the individual curves.

If C is a curve represented as curves

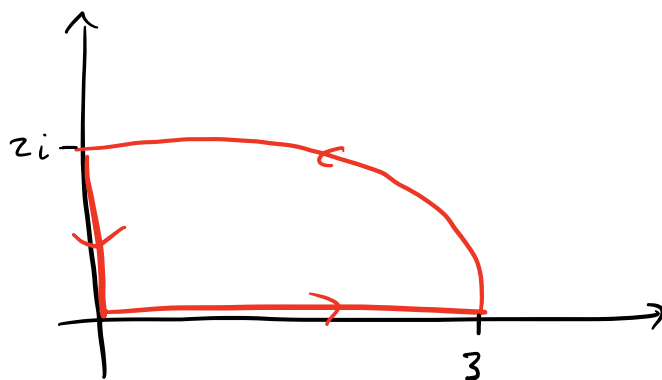
$C_1, C_2, \dots, C_n$ concatenated,



then

$$\int_C f(z) dz = \sum_{k=1}^n \int_{C_k} f(z) dz$$

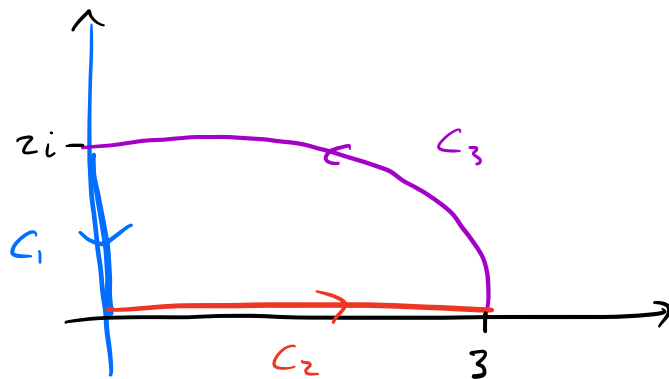
This is helpful when our curve is only piecewise smooth. E.g., suppose C is the curve



where we have a horizontal line segment, a vertical line segment, and an arc of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Let's integrate $\frac{1}{z-(1+i)}$ over this curve.

To do this, we need to parametrize our curve, and the easiest way to do that is to think of the curve as a concatenation of three curves,

say C_1, C_2, C_3 indicated below:



It is easier to parametrize these curves individually. C_1 we can think of as

$$C_1: \quad z(t) = 0 + (2-t)i, \quad 0 \leq t \leq 2$$

C_2 is also easily parametrized,

$$C_2: \quad z(t) = t \quad 0 \leq t \leq 3$$

and C_3 we can parametrize by imagining it's an arc of a circle that's stretched out by a factor of 3 horizontally, and a factor of 2 vertically:

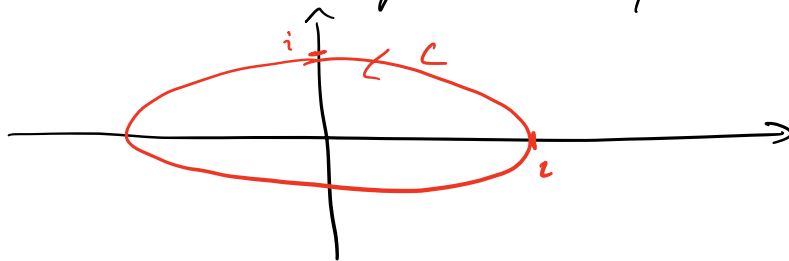
$$C_3: \quad z(t) = 3\cos(t) + i2\sin(t) \quad 0 \leq t \leq \pi/2$$

We can then compute

$$\int_C \frac{1}{z-(1+i)} dz = \int_{C_1} \frac{dz}{z-(1+i)} + \int_{C_2} \frac{dz}{z-(1+i)} + \int_{C_3} \frac{dz}{z-(1+i)}$$

While it is possible to compute these integrals "manually" using our parametrization, it is quite tedious. We will soon see there are some theorems that will greatly simplify this computation.

As another example consider integrating $f(z) = \bar{z}$ over the ellipse $\frac{x^2}{4} + y^2 = 1$,



This is parametrized as $z(t) = 2\cos(t) + i\sin(t)$ for $0 \leq t \leq 2\pi$, so

$$\begin{aligned}
 \int_C \bar{z} dz &= \int_0^{2\pi} (2\cos(t) - i\sin(t)) \cdot (-2\sin(t) + i\cos(t)) dt \\
 &= \int_0^{2\pi} -4\cos(t)\sin(t) + i2\cos^2(t) + 2i\sin^2(t) + \cos(t)\sin(t) dt \\
 &= -3 \int_0^{2\pi} \cos(t)\sin(t) dt + 2i \int_0^{2\pi} dt \\
 &= -3 \int_0^0 u du + 2i \cdot 2\pi \\
 &= 2i \cdot 2\pi
 \end{aligned}$$

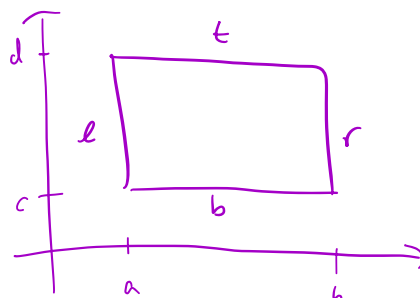
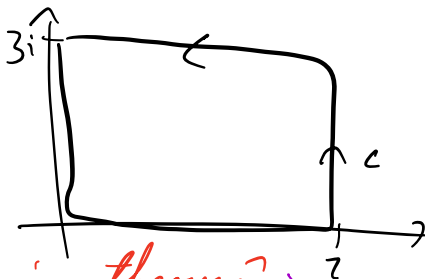
$\swarrow$
 $u = \sin(t)$
 $du = \cos(t) dt$

Outline

- Last time, we left off with the example of

$$\oint_C \bar{z} dz$$

where



- Prove Green's theorem?

- Prove $\oint_C \bar{z} dz = 2i \text{Area}(R)$

OR *Mention Jordan curve theorem*

$$\iint_R (Q_x - P_y) dA = \int_c^d \int_a^b (Q_x - P_y) dx dy$$

$$\oint_{\partial R} P dx + Q dy = \int_b^a P dx + \int_b^a Q dy = \int_b^a P dx$$

- Note if $f(z)$ is a function st. $\int f(z) dz$ depends only on the endpoints of C , then $\oint_C f(z) dz = 0$ for all closed curves C .

- Jordan curve theorem; short, no pf.

- Cauchy's Thm

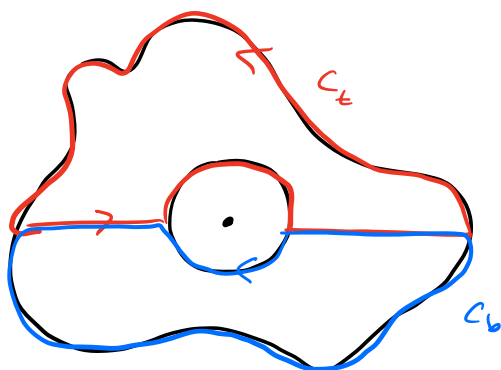
If C is a closed curve and f is holomorphic on the interior of C , then $\oint_C f(z) dz = 0$

Cor

If f is holomorphic, and C, C' are two curves w/ same endpoints in domain of f so that C can be homotoped to C' , keeping endpoints fixed and staying in the domain of f , then $\int_C f(z) dz = \int_{C'} f(z) dz$

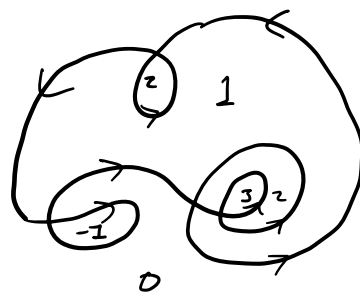
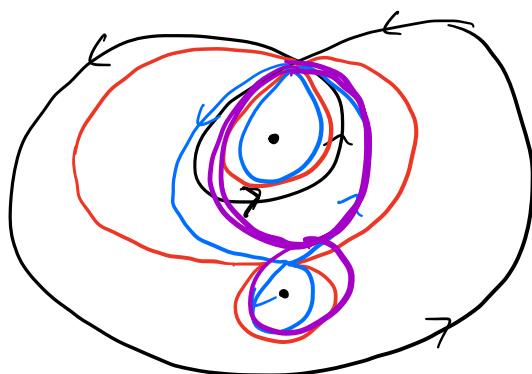
We had previously seen that $\oint_{S'} \frac{dz}{z} = 2\pi i$,

$$\oint_{C_t} \frac{dz}{z} = 0 = \oint_{C_b} \frac{dz}{z}$$



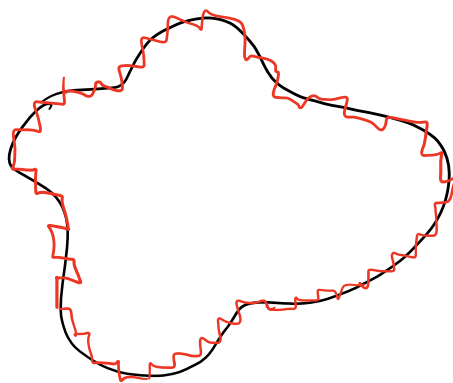
$$0 = \oint_{C_t + C_b} \frac{dz}{z} = \oint_C \frac{dz}{z} - \oint_{S'} \frac{dz}{z}$$

$$\Rightarrow \oint_C \frac{dz}{z} = \oint_{S'} \frac{dz}{z}$$



$$\begin{aligned}
 \iint_S (Q_x - P_y) dA &= \iint_S Q_x dA - \iint_S P_y dA \\
 &= \int_c^d \int_a^b Q_x dx dy - \int_a^b \int_c^d P_y dy dx \\
 &= \int_a^b [Q(b,y) - Q(a,y)] dy
 \end{aligned}$$

$$\oint_{\partial S} Q dy = \int_{\text{right}} Q dy + \int_{\text{left}} Q dy = \int_c^d$$



Green's Theorem

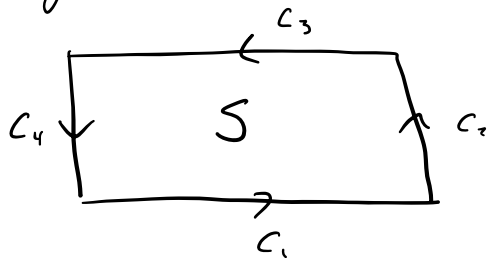
Suppose Q, P are continuously differentiable functions defined in a region S with boundary $\partial S = C$.

Then

$$\oint_C P dx + Q dy = \iint_S (Q_x - P_y) dA$$

Pf

We first suppose S is a rectangle $[a, b] \times [c, d]$ with boundary $C = C_1 \cdot C_2 \cdot C_3 \cdot C_4$ given below



$$C_1: (x(t), y(t)) = (t, c) \quad a \leq t \leq b$$

$$C_2: (x(t), y(t)) = (b, t) \quad c \leq t \leq d$$

$$C_3: (x(t), y(t)) = (a + b - t, d) \quad a \leq t \leq b$$

$$C_4: (x(t), y(t)) = (a, c + d - t) \quad c \leq t \leq d$$

We can verify Green's theorem by direct computation:

$$\begin{aligned} \iint_S (Q_x - P_y) dA &= \iint_S Q_x dA - \iint_S P_y dA \\ &= \int_c^d \int_a^b Q_x dx dy - \int_a^b \int_c^d P_y dy dx \\ &= \int_c^d [Q(b, y) - Q(a, y)] dy - \int_a^b [P(x, c) - P(x, d)] dx \end{aligned}$$

Now we compute the line integrals,

$$\oint_C Pdx + Qdy = \int_{C_1} Pdx + Qdy + \int_{C_2} Pdx + Qdy + \int_{C_3} Pdx + Qdy + \int_{C_4} Pdx + Qdy$$

Note $dy=0$ on C_1 and C_3 ; $dx=0$ on C_2 and C_4 , so this simplifies to

$$\int_{C_1} Pdx + \int_{C_2} Qdy + \int_{C_3} Pdx + \int_{C_4} Qdy$$

Using our parametrizations we have

$$\int_{C_1} Pdx = \int_a^b P(t, c) dt = \int_a^b P(x, c) dx$$

$$\int_{C_2} Qdy = \int_c^d Q(b, t) dt = \int_c^d Q(b, y) dy$$

$$\int_{C_3} Pdx = \int_a^b P(a+b-t, d) \cdot \overset{x'(t)=-1}{\downarrow} (-1) dt$$

Letting $u = a+b-t$, $du = -dt$ this becomes

$$\int_b^a P(u, d) du = - \int_a^b P(u, d) du = - \int_a^b P(x, d) dx$$

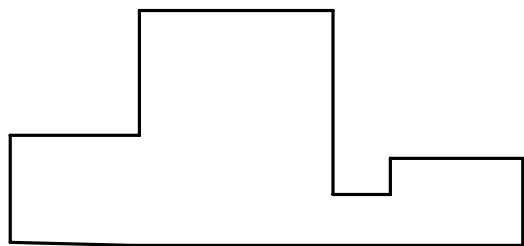
$$\int_{C_4} Qdy = \int_c^d Q(a, c+d-t) (-1) dt$$

Using $u = c + d - t$, $du = -dt$ this becomes

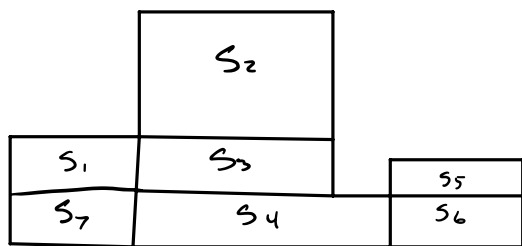
$$\int_d^c Q(a, u) du = - \int_c^d Q(a, u) du = - \int_c^d Q(a, y) dy$$

Adding these together gives us the same expression as our earlier double integral, so Green's theorem is verified for rectangles.

For "rectilinear" regions whose boundary consists of horizontal & vertical segments.



We decompose the region into rectangles,



Note

$$\oint_{\partial S} P dx + Q dy = \sum_{k=1}^n \oint_{\partial S_k} P dx + Q dy$$

as overlapping interior boundaries cancel out



Also note for the double integral,

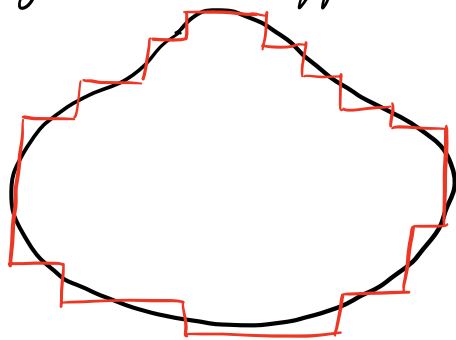
$$\iint_S (Q_x - P_y) dA = \sum_{k=1}^n \iint_{S_k} (Q_x - P_y) dA$$

But each S_k is a rectangle, so

$$\iint_{S_k} (Q_x - P_y) dA = \oint_{\partial S_k} P dx + Q dy$$

Thus Green's theorem holds for rectilinear regions.

For general regions, we approximate w/ rectilinear regions,



R = general region
 S = rectilinear region

For every $\epsilon > 0$ we can find a rectilinear S approximating R so that

$$\left| \iint_R (Q_x - P_y) dA - \iint_S (Q_x - P_y) dA \right| < \epsilon/2 \quad \text{and}$$

$$\left| \oint_{\partial R} P dx + Q dy - \oint_{\partial S} P dx + Q dy \right| < \epsilon/2$$

Thus

$$\begin{aligned} & \left| \oint_{\partial R} P dx + Q dy - \iint_R (Q_x - P_y) dA \right| \\ &= \left| \oint_{\partial R} P dx + Q dy - \oint_{\partial S} P dx + Q dy + \iint_S (Q_x - P_y) dA - \iint_R (Q_x - P_y) dA \right| \\ &\leq \left| \oint_{\partial R} P dx + Q dy - \oint_{\partial S} P dx + Q dy \right| + \left| \iint_S (Q_x - P_y) dA - \iint_R (Q_x - P_y) dA \right| \\ &\leq \varepsilon/2 + \varepsilon/2 \end{aligned}$$

Since this holds for all $\varepsilon > 0$,

$$\oint_{\partial R} P dx + Q dy = \iint_R (Q_x - P_y) dA.$$

□

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Lecture 19 - Complex Integration, pt. 3

In the last lecture we had observed that if f is holomorphic in an open set U and if C is a closed curve such that both C and its interior remain inside of U , then

$$\oint_C f(z) dz = 0 \quad (\text{This is Cauchy's theorem})$$

As a consequence, if a curve C can be continuously deformed into a curve C' with the deformation preserving the endpoints of C and remaining in the set U , then

$$\int_C f(z) dz = \int_{C'} f(z) dz$$

We used this to show that for any curve C encircling the origin exactly once with positive (counter-clockwise) orientation, we must have

$$\int_C \frac{1}{z} dz = 2\pi i$$

Let's first extend the proof of that result to obtain a more general theorem

Thm (The Deformation Theorem)

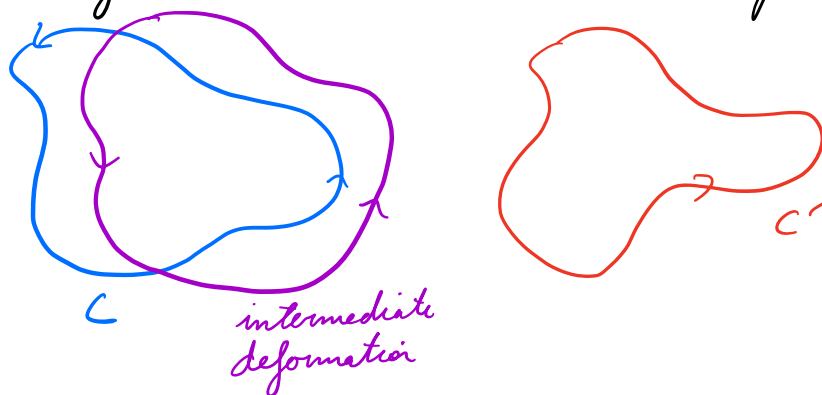
Suppose $U \subseteq \mathbb{C}$ is open and $f: U \rightarrow \mathbb{C}$ is holomorphic. If C, C' are any two closed curves in U where C can be continuously deformed to C' with the deformation remaining in U , then

$$\oint_C f(z) dz = \oint_{C'} f(z) dz$$

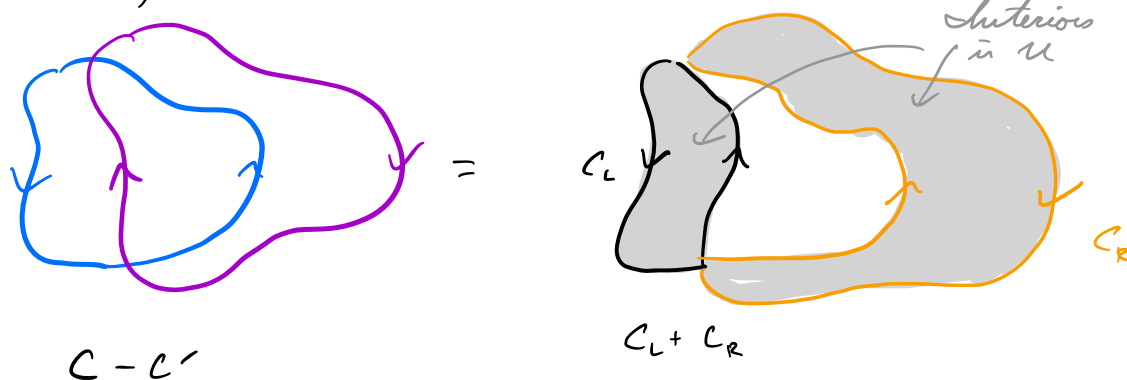
Before giving the proof of the deformation theorem, observe that unlike in Cauchy's theorem we are not assuming f remains holomorphic in the interior of C , and so we do not necessarily have $\oint_C f(z) dz = 0 = \oint_{C'} f(z) dz$.

Pf

We will first give the intuition of the proof, then make the proof precise. The idea is that as we deform C into C' , the difference between the initial C and some intermediate deformation is a closed loop inside U ,



Note that $\oint_C f(z) dz = \oint_{C'} f(z) dz$ as their difference is made up of closed loops which are completely contained in U , as are their interiors



$$\oint_C f(z) dz - \oint_{C'} f(z) dz = \oint_{C_L} f(z) dz + \oint_{C_R} f(z) dz = 0$$

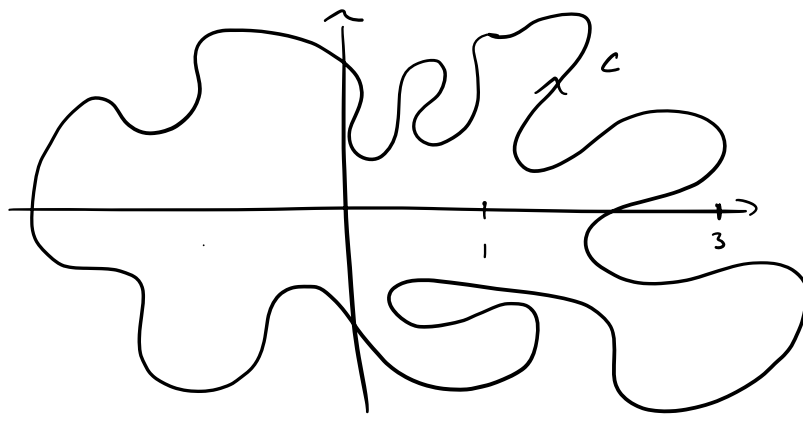
Thus the difference between "nearby" deformations is always zero. □

The main take away from the deformation theorem is that as long as we stay inside of the domain of our holomorphic function while deforming, we don't risk changing the value of our integral, so we can replace a curve with an easier to parametrize curve when doing calculations.

Example

Compute $\oint_C \frac{dz}{z^2 - 1}$ where C is the curve indicated below:

↖ Change to $\frac{1}{z(z-1)(z-3)}$



By the deformation theorem, this equals $\oint_{C'} \frac{1}{z^2-1} dz$ where C' is, say, a circle of radius 2 centered at the origin. Thus we can compute

$$\begin{aligned} \oint_C \frac{1}{z^2-1} dz &= \oint_{\partial B_2(0)} \frac{1}{z^2-1} dz \\ &= \int_0^{2\pi} \frac{1}{(2\cos t + i2\sin t)^2 - 1} \cdot (-2\sin t + i2\cos t) dt \\ &= \int_0^{2\pi} \frac{-2\sin t + i2\cos t}{4\cos^2 t + 8i\cos t \sin t - 4\sin^2 t} dt \end{aligned}$$

Let's notice that it's very important our curves stay w/in the domain of a holomorphic function as we deform the curve — if we "skip over" a point where the function is undefined, the integral can instantaneously change.

E.g., compare the values of $\int_{C_1} \frac{dz}{z} \sim \int_{C_2} \frac{dz}{z}$ where C_1 is a curve connecting ± 1 in the upper half plane and C_2 is a curve connecting the same points in the lower half plane.

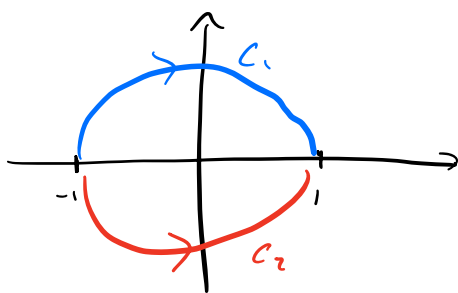
$$\int_{C_1} \frac{dz}{z}$$

$$= \int_{\pi}^{2\pi} \frac{\sin(-t) - i\cos(-t)}{\cos(-t) + i\sin(-t)} dt$$

$$= \int_{\pi}^{2\pi} [\sin(t) - i\cos(t)] \cdot [\cos(t) - i\sin(t)] dt$$

$$= \int_{\pi}^{2\pi} -i dt$$

$$= -\pi i$$



$$\int_{C_2} \frac{dz}{z}$$

$$= \int_{\pi}^{2\pi} \frac{-\sin(t) + i\cos(t)}{\cos(t) + i\sin(t)} dt$$

$$= \int_{\pi}^{2\pi} [-\sin(t) + i\cos(t)] \cdot [\cos(t) - i\sin(t)] dt$$

$$= \int_{\pi}^{2\pi} i dt$$

$$= \pi i$$

Before proving our main result for today, we need the following lemma, which we can view as a mild generalization of Cauchy's theorem

Thm

Suppose C is a closed curve with interior U and let z_0 be a fixed point in U . If f is a function defined on a neighborhood of U which is continuous everywhere it's defined and is holomorphic at every point of U except possibly z_0 , then

$$\oint_C f(z) dz = 0$$

Pf By the deformation theorem, for every $r > 0$ so that $B_r(z_0) \subseteq U$, we have

$$\oint_C f(z) dz = \oint_{\partial B_r(z_0)} f(z) dz$$

Let R be some fixed choice of r - i.e., $B_R(z_0) \subseteq U$.

Since f is continuous, $|f|$ is also continuous. Thus there exists some constant M so that $|f(z)| \leq M$ for all $z \in B_R(z_0)$. So, for each $0 < r \leq R$ we have

$$\left| \oint_C f(z) dz \right| = \left| \oint_{\partial B_r(z_0)} f(z) dz \right| \leq \oint_{\partial B_r(z_0)} |f(z)| dz \leq \oint_{\partial B_r} M dz = M 2\pi r$$

So, for each $0 < r \leq R$, we have

$$\left| \oint_{\gamma} f(z) dz \right| \leq M 2\pi r$$

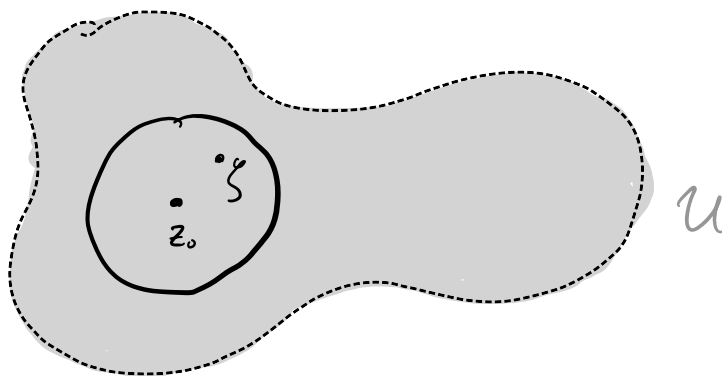
Letting r decrease to zero shows $\left| \oint_{\gamma} f(z) dz \right| = 0$,
so $\oint_{\gamma} f(z) dz = 0$. □

With the above, we can now prove the following amazing theorem that allows us to express the value of a holomorphic function f at a point as an integral around that point.

Thm (Cauchy Integral Formula)

Suppose $f: U \rightarrow \mathbb{C}$ is holomorphic, $U \subseteq \mathbb{C}$ is open,
Let z_0 be any point in U and $r > 0$
chosen so $B_r(z_0) \subseteq U$. For every $\xi \in B_r(z_0)$,

$$f(\xi) = \frac{1}{2\pi i} \oint_{\partial B_r(z_0)} \frac{f(z)}{z - \xi} dz$$



Pf Let $g: U \rightarrow \mathbb{C}$ be the function

$$g(z) = \begin{cases} \frac{f(z) - f(\xi)}{z - \xi} & \text{if } z \neq \xi \\ f'(\xi) & \text{if } z = \xi \end{cases}$$

Note g is holomorphic on $U \setminus \{\xi\}$, and continuous on all of U . Thus

$$\oint_{\partial B_r(z_0)} g(z) dz = \oint_{\partial B_r(z_0)} \frac{f(z) - f(\xi)}{z - \xi} dz = 0$$

Now we write

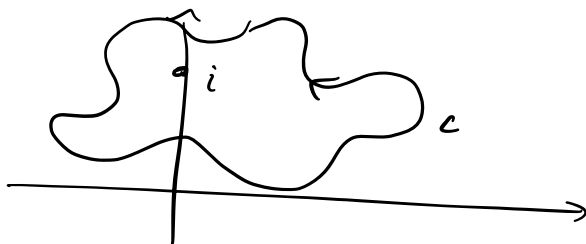
$$\oint_{\partial B_r(z_0)} \frac{f(z) - f(\xi)}{z - \xi} dz = \oint_{\partial B_r(z_0)} \frac{f(z)}{z - \xi} dz - \oint_{\partial B_r(z_0)} \frac{f(\xi)}{z - \xi} dz$$

$$\Rightarrow \oint_{\partial B_r(z_0)} \frac{f(z)}{z - \xi} dz = f(\xi) \cdot \oint_{\partial B_r(z_0)} \frac{dz}{z - \xi} = 2\pi i f(\xi). \quad \square$$

We'll use this to prove some important theorems later, but notice this sometimes gives us a way to replace complicated integrals by simple function evaluation.

Ex

Compute $\oint_C \frac{z^2 + 2z + 5}{z - i} dz$ where C is the curve below,



Letting $f(z) = z^2 + 2z + 5$, Cauchy's integral formula and the deformation theorem tell us

$$f(i) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - i} dz$$

$$\Rightarrow i^2 + 2i + 5 = \frac{1}{2\pi i} \oint_C \frac{z^2 + 2z + 5}{z - i} dz$$

$$\Rightarrow \oint_C \frac{z^2 + 2z + 5}{z - i} dz = 2\pi i (i^2 + 2i + 5) = 2\pi i (4 + 2i) \\ = -4\pi + 8\pi i$$

We can also read off the value of derivative $f^{(n)}(\xi)$ from the integral

Thm (Cauchy's Integral Formula for Derivatives)

Suppose $f: U \rightarrow \mathbb{C}$ is holomorphic, $U \subseteq \mathbb{C}$ is open,

Let z_0 be any point in U and $r > 0$ chosen so $B_r(z_0) \subseteq U$. For every $\xi \in B_r(z_0)$,

$$f^{(n)}(\xi) = \frac{n!}{2\pi i} \oint_{\partial B_r(z_0)} \frac{f(z)}{(z - \xi)^{n+1}} dz$$

Pf We simply differentiate both sides of the equation provided by Cauchy's integral formula with respect to ζ .

$$f(\zeta) = \frac{1}{2\pi i} \oint_{\partial B_r(z_0)} \frac{f(z)}{z - \zeta} dz$$

$$\Rightarrow \frac{d}{d\zeta} f(\zeta) = \frac{d}{d\zeta} \frac{1}{2\pi i} \oint_{\partial B_r(z_0)} f(z) \cdot [z - \zeta]^{-1} dz$$

$$= \frac{1}{2\pi i} \oint_{\partial B_r(z_0)} \frac{d}{d\zeta} f(z) \cdot [z - \zeta]^{-1} dz$$

$$\Rightarrow f'(\zeta) = \frac{1}{2\pi i} \oint_{\partial B_r(z_0)} \frac{f(z)}{(z - \zeta)^2} dz$$

This establishes the theorem when $n=1$. For $n>1$ we proceed by induction. Suppose we have established

$$f^{(n-1)}(\zeta) = \frac{(n-1)!}{2\pi i} \oint_{\partial B_r(z_0)} \frac{f(z)}{(z - \zeta)^n} dz$$

Then

$$\begin{aligned} f^{(n)}(\zeta) &= \frac{d}{d\zeta} \frac{(n-1)!}{2\pi i} \oint_{\partial B_r(z_0)} \frac{f(z)}{(z - \zeta)^n} dz \\ &= \frac{(n-1)!}{2\pi i} \oint_{\partial B_r(z_0)} \frac{d}{d\zeta} f(z) \cdot (z - \zeta)^{-n} dz \\ &= \frac{n!}{2\pi i} \oint_{\partial B_r(z_0)} \frac{f(z)}{(z - \zeta)^{n+1}} dz \end{aligned}$$

□

Ex

Compute $\oint_{S'} \frac{e^{2z}}{z^4} dz$

Letting $f(z) = e^{2z}$, note $f'(z) = 2e^{2z}$, $f''(z) = 4e^{2z}$
and $f'''(z) = 8e^{2z}$. Then

$$f^{(3)}(0) = \frac{3!}{2\pi i} \oint_{S'} \frac{e^{2z}}{z^4} dz$$

$$\Rightarrow \oint_{S'} \frac{e^{2z}}{z^4} dz = \frac{8 \cdot 2\pi i}{6} = \frac{8\pi i}{3}$$

We now have all the tools necessary to prove the theorem we've mentioned before: if a function is holomorphic, then it can be represented as a power series locally.

Thm

Suppose $f: U \rightarrow \mathbb{C}$ is holomorphic. For each $z_0 \in U$, f can be represented as a power series centered at z_0 converging in each ball $B_r(z_0)$ such that $\overline{B_r(z_0)} \subseteq U$.

Pf

Let ξ be any point in $B_r(z_0)$. Then

$$f(\xi) = \frac{1}{2\pi i} \oint_{\partial B_r(z_0)} \frac{f(z)}{z - \xi} dz$$

$$= \frac{1}{2\pi i} \oint_{\partial B_r(z_0)} \frac{1}{z - z_0 + z_0 - \zeta} f(z) dz$$

$$= \frac{1}{2\pi i} \oint_{\partial B_r(z_0)} \frac{1}{z - z_0 - (\zeta - z_0)} f(z) dz$$

$$= \frac{1}{2\pi i} \oint_{\partial B_r(z_0)} \frac{1}{(z - z_0) \cdot \left(1 - \frac{\zeta - z_0}{z - z_0}\right)} f(z) dz$$

$$= \frac{1}{2\pi i} \oint_{\partial B_r(z_0)} \frac{1}{z - z_0} \cdot \sum_{n=0}^{\infty} \left(\frac{\zeta - z_0}{z - z_0}\right)^n f(z) dz$$

$$= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{\partial B_r(z_0)} \frac{(\zeta - z_0)^n}{(z - z_0)^{n+1}} f(z) dz$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \oint_{\partial B_r(z_0)} \frac{f(z)}{(z - z_0)^{n+1}} dz \right] (\zeta - z_0)^n$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (\zeta - z_0)^n$$

□

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Lecture 20 - Liouville's Theorem, Fundamental Thm of Algebra

In this lecture we'll use some of the theory we've developed to prove a few particularly important theorems.

We'll begin, though, with a very simple definition: We call a function $f: \mathbb{C} \rightarrow \mathbb{C}$ entire if it is holomorphic at every point of the complex plane.

Some easy examples of entire functions are polynomials, sine and cosine, and the exponential function e^z .

One key property of entire functions is that they never attain a max or min — in fact, they aren't even bounded: if f is entire, then for every $M \in \mathbb{R}$, $\exists z \in \mathbb{C}$ so that $|f(z)| > M$. This result is called "Liouville's theorem," and we'll prove it in just a moment. First, though, notice there is no analogue of Liouville's theorem in the real setting. E.g., $\arctan(x)$ is defined on all of $\mathbb{R}$ but is bounded, $-\frac{\pi}{2} < \arctan(x) < \frac{\pi}{2}$.

Theorem (Liouville's theorem)

Pf The only bounded entire functions are constants.

Suppose $f: \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic and bounded,
so $\exists M \in \mathbb{R}$ such that $\forall z \in \mathbb{C}, |f(z)| < M$.

We will show f is constant by showing its derivative is zero at every point.

Using Cauchy's integral formula for derivative,
for each $\xi \in \mathbb{C}$ and every $r > 0$, we have

$$f'(\xi) = \frac{1}{2\pi i} \oint_{\partial B_r(\xi)} \frac{f(z)}{(z-\xi)^2} dz$$

$$\Rightarrow |f'(\xi)| = \left| \frac{1}{2\pi i} \oint_{\partial B_r(\xi)} \frac{f(z)}{(z-\xi)^2} dz \right|$$

$$\leq \frac{1}{2\pi} \oint_{\partial B_r(\xi)} \left| \frac{f(z)}{(z-\xi)^2} \right| dz$$

$$= \frac{1}{2\pi} \oint_{\partial B_r(\xi)} \frac{|f(z)|}{|z-\xi|^2} dz$$

$$\leq \frac{1}{2\pi} \oint_{\partial B_r(\xi)} \frac{M}{r^2} dz$$

$$= \frac{1}{2\pi} \cdot 2\pi i \cdot \frac{M}{r^2} = \frac{M}{r}$$

So, for every $r > 0$, $|f'(\xi)| < \frac{M}{r}$. Letting r become arbitrarily large, $|f'(\xi)|$ is arbitrarily small, so $f'(\xi) = 0$. □

With Liouville's theorem at our disposal, we can now prove one of the most important theorems in all of mathematics.

Thm (Fundamental Theorem of Algebra)

Every polynomial with complex coefficients has a root.

Pf

Let $f(z) = a_n z^n + \dots + a_1 z + a_0$ be a polynomial with complex coefficients. We first show that if z is "sufficiently large" in absolute value, then $f(z)$ can be made arbitrarily large in absolute value as well. That is, given $M > 0$ there exists an $r > 0$ so that if $|z| > r$, then $|f(z)| > M$. (Homework exercise.) Now consider the function $1/f(z)$. If $f(z)$ has no roots, it is an entire function. But for large z , $1/f(z)$ is bounded:

$$\Rightarrow |1/f(z)| < 1/M$$

$$\Rightarrow \frac{1}{f(z)} < \frac{1}{M}$$

If $|z| < r$, then $|1/f(z)|$ attains a maximum

by the extreme value theorem. Thus $1/f(z)$ is a bounded entire function, but by Liouville's theorem this is impossible. $\square$

Corollary

Every polynomial with complex coefficients may be written uniquely (up to permutation of the f_j)

$$c(z-f_1)^{m_1}(z-f_2)^{m_2}\dots(z-f_k)^{m_k}$$

where $f_j \in \mathbb{C}$, $m_j \in \mathbb{N}$, and $\sum_{j=1}^k m_j$ equals the degree of the polynomial.

Pf

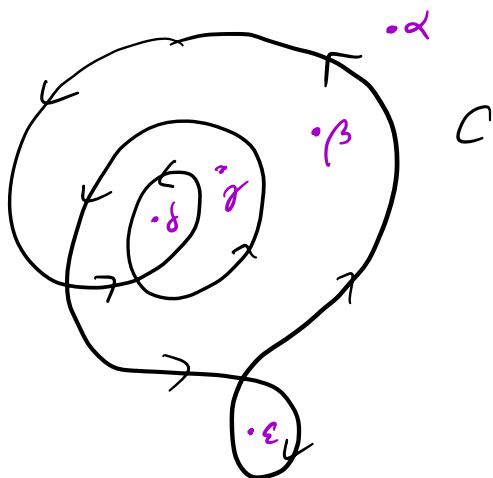
Homework. $\square$

To continue our study of holomorphic functions and complex integration, it will be helpful to make the following definition. Given a closed curve C and a point $\zeta \notin C$, we define the winding number (also called the index) of C around ζ as

$$n(C; \zeta) = \frac{1}{2\pi i} \int_C \frac{dz}{z - \zeta}$$

This number "counts" the number of times

C winds around ζ counter-clockwise



$$n(C; \alpha) = 0$$

$$n(C; \beta) = 1$$

$$n(C; \gamma) = 2$$

$$n(C; \delta) = 3$$

$$n(C; \epsilon) = -1$$

As we'll soon see, many results can be expressed in terms of winding numbers. For example, we can extend Cauchy's integral formula to curves which are not simple by incorporating the winding number

Thm (Cauchy integral formula for non-simple curves)

Let $f: U \rightarrow \mathbb{C}$ be holomorphic function and C a curve contained in U where $n(C; z) = 0$ for all $z \notin U$. Then for each $\zeta \in U \setminus C$,

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z - \zeta} dz = f(\zeta) \cdot n(C; \zeta)$$

Pf

Homework. (Hint: If C is a simple closed curve with positive orientation, this is simply the version of Cauchy's integral formula we

proved in the last lecture. If C is simple with negative orientation, we just negate the integral. If C is not simple, we can use the deformation theorem and imagine C as a concatenation of s.c.c.'s.)

We say a point a in the domain of a holomorphic f is a zero of multiplicity m if the power series representation of f centered at a has the form

$$\sum_{k=m}^{\infty} c_k (z-a)^k = c_m (z-a)^m + c_{m+1} (z-a)^{m+1} + \dots$$

with $c_m \neq 0$. Note in this case we can factor $(z-a)^m$ out to obtain

$$f(z) = (z-a)^m g(z)$$

where g is a holomorphic function and $g(a) \neq 0$.

Iterating, if f has zeros $a_1, a_2, \dots, a_n$ of respective multiplicities $m_1, \dots, m_n$, we can write

$$f(z) = (z-a_1)^{m_1} \cdot (z-a_2)^{m_2} \cdot \dots \cdot (z-a_n)^{m_n} g(z)$$

where g is holomorphic and not zero at any a_k .

Notice

$$\begin{aligned}
 f'(z) &= m_1 (z-a_1)^{m_1-1} \cdot (z-a_2)^{m_2} \cdot (z-a_3)^{m_3} \cdots (z-a_n)^{m_n} g(z) \\
 &+ (z-a_1)^{m_1} \cdot m_2 (z-a_2)^{m_2-1} \cdot (z-a_3)^{m_3} \cdots (z-a_n)^{m_n} g(z) \\
 &+ \cdots \\
 &+ (z-a_1)^{m_1} (z-a_2)^{m_2} \cdots m_n (z-a_n)^{m_n-1} g(z) \\
 &+ (z-a_1)^{m_1} \cdot (z-a_2)^{m_2} \cdots (z-a_n)^{m_n} g'(z)
 \end{aligned}$$

Dividing this by $f(z)$ gives

$$\frac{f'(z)}{f(z)} = \frac{m_1}{z-a_1} + \frac{m_2}{z-a_2} + \cdots + \frac{m_n}{z-a_n} + \frac{g'(z)}{g(z)}$$

Now let's notice if C is a closed curve encircling the a_k , we have

$$\begin{aligned}
 \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz &= \frac{1}{2\pi i} \int_C \left(\frac{m_1}{z-a_1} + \frac{m_2}{z-a_2} + \cdots + \frac{m_n}{z-a_n} + \frac{g'(z)}{g(z)} \right) dz \\
 &= \frac{m_1}{2\pi i} \int_C \frac{dz}{z-a_1} + \frac{m_2}{2\pi i} \int_C \frac{dz}{z-a_2} + \cdots + \frac{m_n}{2\pi i} \int_C \frac{dz}{z-a_n} + \frac{1}{2\pi i} \int_C \frac{g'(z)}{g(z)} dz \\
 &= \sum_{k=1}^n m_k n(C, a_k)
 \end{aligned}$$

Thus $\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz$ counts the number of zeros, by multiplicity and winding number,

enclosed by C .

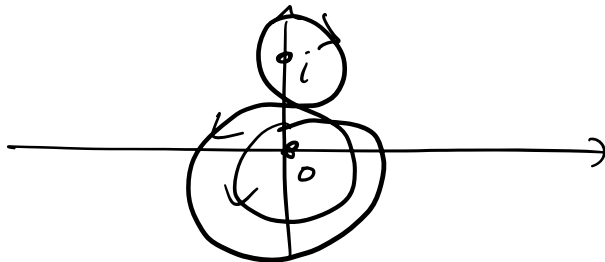
E.g., Compute $\oint_C \frac{3z^2 + 2z + 1}{z^3 + z^2 + z + 1} dz$ where C is the circle of radius 2 centered at the origin.

Note our integral has the form $\oint_C \frac{f'(z)}{f(z)} dz$ where

$$\begin{aligned} f(z) &= z^3 + z^2 + z + 1 = z^2(z+1) + z+1 \\ &= (z^2+1)(z+1) \\ &= (z+i)(z-i)(z+1) \end{aligned}$$

There three zeros of multiplicity 1 are enclosed by C , and so our integral is $2\pi i(1+1+1) = 6\pi i$.

As another example, let's compute $\int_C \frac{3z^2 - 2zi}{6z^3 - 6z^2i} dz$ where C is the curve below



By our earlier calculation, taking

$$f(z) = 6z^3 - 6z^2i = 6z^2(z-i)$$

we have

$$\frac{1}{2\pi i} \int_C \frac{18z^2 - 12zi}{6z^3 - 6z^2i} dz = 2 \cdot 2 - 1 = 3$$

$$\Rightarrow \frac{6}{2\pi i} \int_C \frac{3z^2 - 2zi}{6z^3 - 6z^2i} dz = 3$$

$$\Rightarrow \int_C \frac{3z^2 - 2zi}{6z^3 - 6z^2i} dz = \pi i$$

Outline

- Basic motivation
- Define "isolated points"
 - Use identity theorem to show solution to $f(z) = z_0$ is isolated if f is non-const. holomorphic function
- Define singularities
 - Removable singularity Example: $\frac{z^2+1}{z-i}$
 - Pole Example: $1/z$
 - Essential singularities Example?
- Then about reaching any value

Contradict theorem!

Maybe not, but good exercise for the students

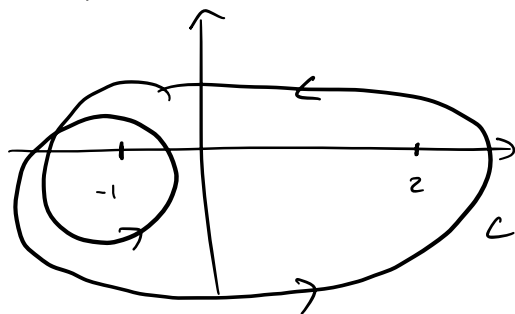
Math 424, Spring 2024

Lecture 21 - Singularities, Meromorphic Functions, and the Riemann Sphere

In studying integrals of complex-valued functions, we saw, by Cauchy's theorem, the integral of a function around a closed curve C , $\oint_C f(z) dz$, is zero if f is holomorphic on the interior of C . For many integrals we've considered, though, the function isn't holomorphic inside all of the region bounded by C , but instead may have points where the function is undefined.

E.g., the value of $\int_C \frac{1}{z^2 - z - 2} dz = \int_C \frac{1}{(z+1)^2(z-2)^2} dz$

where C is the curve below,



$$\begin{aligned} & (z^2 - z - 2)(z^2 - z - 2) \\ &= z^4 - z^3 - 2z^2 - z^3 + z^2 + 2z \\ & \quad - 2z^2 + 2z + 4 \\ &= z^4 - 2z^3 + 3z^2 + 4z + 4 \end{aligned}$$

should not be expected to vanish since the function is not holomorphic inside the curve

(Exercise: Compute the value of the integral above.)

We'll now start to study these "singularities" of otherwise holomorphic functions, and will later see how certain values associated to the singularities determine the value of integrals around closed curves.

First we'll make a topological definition to simplify our language a little. We will say a set $S \subseteq \mathbb{C}$ is isolated (or that the points of S are isolated points) if for each $x \in S$ there exists an open set U_x with $x \in U_x$ and $y \notin U_x$ for all other $y \in S \setminus \{x\}$.

For example, $S = \mathbb{Z}$ is an isolated set since for each $x \in \mathbb{Z}$ the set $U_x = B_{1/4}(x)$ will be an open neighborhood avoiding the rest of $\mathbb{Z}$



The set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ is isolated, but $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is not.

(Exercise: What are the U_x sets for $\{\frac{1}{n} \mid n \in \mathbb{N}\}$? Why is $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ not isolated?)

Now suppose that U is an open set and $S \subseteq U$ is a subset of isolated points. Given a function $f: U \setminus S \rightarrow \mathbb{C}$ which is holomorphic, we call the points of S singularities of f . For example, the function

$$f(z) = \frac{z^2 + 1}{z^2 - iz}$$

is holomorphic on $\mathbb{C} \setminus \{0, i\}$, and so 0 and i are singularities of this function.

It will turn out that singularities come in a few particular types.

A singularity z_0 of a holomorphic function f is called removable if $\lim_{z \rightarrow z_0} f(z)$ exists.

In this case we can extend f to a holomorphic function $\tilde{f}$ defined at z_0 by declaring $\tilde{f}(z) = f(z)$ if $z \neq z_0$, and $\tilde{f}(z_0) = \lim_{z \rightarrow z_0} f(z)$. Often we'll just "fill in" the removable singularity and replace f by $\tilde{f}$.

Our function $f(z) = \frac{z^2 + 1}{z^2 - iz}$ above has a removable singularity at i as

$$\lim_{z \rightarrow i} f(z) = \lim_{z \rightarrow i} \frac{z^2 + 1}{z^2 - iz} = \lim_{z \rightarrow i} \frac{(z+i)(z-i)}{z(z-i)} = \frac{2i}{i} = 2$$

We will say a singularity z_0 is non-essential if there exists an integer m so that the function $g(z) = (z - z_0)^m f(z)$ has a removable singularity at z_0 . Of course removable singularities are non-essential by definition (take $m=0$), but some non-removable singularities can also be non-essential.

In our function above,

$$f(z) = \frac{z^2 + 1}{z^2 - iz}$$

There is a non-essential singularity at 0, as

$$\lim_{z \rightarrow 0} z \cdot \frac{z^2 + 1}{z^2 - iz} = \lim_{z \rightarrow 0} z \cdot \frac{z^2 + 1}{z(z - i)} = \lim_{z \rightarrow 0} \frac{z^2 + 1}{z - i} = \frac{1}{-i} = i$$

Notice, though, that 0 is not a removable singularity as

$$\begin{aligned} \lim_{z \rightarrow 0} |f(z)| &= \lim_{z \rightarrow 0} \left| \frac{z^2 + 1}{z^2 - iz} \right| = \lim_{z \rightarrow 0} \frac{|z + i|}{|z|} = \lim_{z \rightarrow 0} \frac{1}{|z|} \cdot |z + i| \\ &= \infty \end{aligned}$$

In general, non-essential singularities which are non-removable are called poles and necessarily have the property that $\lim_{z \rightarrow z_0} |f(z)| = \infty$. (Exercise)

Given a holomorphic function $f: U \setminus \{z_0\} \rightarrow \mathbb{C}$ which has a non-essential singularity z_0 , we define the order of f at z_0 , denoted $\text{ord}(f; z_0)$, to be the negative of the minimum value m so that $(z - z_0)^m f(z)$ has a removable singularity at z_0 .

For example, in our running example of $f(z) = \frac{z^2 + 1}{z - iz}$, the singularities of i and 0 have orders 1 and -1 , respectively. As another example, consider

$$f(z) = \frac{z^3 - 3z + 2}{z^6 + 3z^5 - 4z^2} = \frac{(z-1)^2(z+2)}{z^3 \cdot (z+2)^2(z-1)}$$

This function has singularities at $0, -2$, and 1 and the orders of these singularities are

$$\text{ord}(f; 0) = -3, \quad \text{ord}(f; -2) = -1, \quad \text{ord}(f; 1) = 1$$

$\uparrow$ 0 is a triple pole

$\uparrow$ -2 is a simple pole

$\uparrow$ 1 is a simple zero

Then

Suppose $f: U \setminus \{z_0\} \rightarrow \mathbb{C}$ is holomorphic with a non-essential singularity at z_0 . Then

- $\text{ord}(f; z_0) \geq 0$ if and only if z_0 is a removable singularity
- $\text{ord}(f; z_0) = 0$ if and only if z_0 is a removable singularity and $f(z_0) \neq 0$ (after "extending" the function to z_0)
- $\text{ord}(f; z_0) < 0$ if and only if z_0 is a pole.

Pf
Exercise

If f has a pole at z_0 and $\text{ord}(f; z_0) = m < 0$, we sometimes say z_0 is a pole of order $-m$. For example, if $f(z) = \frac{1}{z^3}$, then $\text{ord}(f; 0) = -3$ and we say 0 is a pole of order 3. Poles of order 1 are also referred to as simple poles; poles of order 2 are double poles; poles of order 3 are triple poles; etc.

Thm

An isolated singularity z_0 of a function f is removable if and only if $\lim_{z \rightarrow z_0} (z - z_0) f(z) = 0$.

Pf

If z_0 is removable, $\lim_{z \rightarrow z_0} f(z) = \zeta \in \mathbb{C}$, so $\lim_{z \rightarrow z_0} (z - z_0) f(z) = 0 \cdot \zeta = 0$.

If $\lim_{z \rightarrow z_0} (z - z_0) f(z) = 0$, then $(z - z_0) f(z)$ holomorphic, so $(z - z_0) f(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \dots$

But this equals 0 at z_0 , so $c_0 = 0$ and we have

$$(z - z_0) f(z) = c_1(z - z_0) + c_2(z - z_0)^2 + c_3(z - z_0)^3 + \dots$$

$$\Rightarrow f(z) = c_1 + c_2(z - z_0) + c_3(z - z_0)^2 + \dots \quad \text{for } z \neq z_0$$

But the limit of this as $z \rightarrow z_0$ is c_1 , so $\lim_{z \rightarrow z_0} f(z) = c_1$, and z_0 is a removable singularity. □

Thm (Riemann's removability criterion)

An isolated singularity z_0 of a holomorphic $f: U \setminus \{z_0\} \rightarrow \mathbb{C}$ is removable if and only if there exists an $\epsilon > 0$ so that the punctured disc

$$\dot{B}_r(z_0) := B_r(z_0) \setminus \{z_0\}$$

is contained in the domain of f and f is bounded on this punctured disc.

Pf

If z_0 is removable, the condition is satisfied as $\lim_{z \rightarrow z_0} f(z)$ is some finite value, say $\xi \in \mathbb{C}$.

Then $\forall \epsilon > 0, \exists \delta > 0$ so that $0 < |z - z_0| < \delta$ implies $|f(z) - \xi| < \epsilon$. Thus $f(z) \in B_\epsilon(\xi)$, and so is bounded.

Now suppose $f(z)$ is bounded for $z \in \dot{B}_r(z_0)$. Then $\lim_{z \rightarrow z_0} (z - z_0) \cdot f(z) = 0$, and $f(z)$ has a removable singularity at z_0 by our earlier theorem. □

Note something akin to the Riemann removability criterion is very false in the real setting. In particular consider $\sin(\frac{1}{x})$ which is undefined at $x=0$, but is bounded. However $x=0$ is not a removable singularity of $\sin(\frac{1}{x})$ as $\lim_{x \rightarrow 0} \sin(\frac{1}{x})$ DNE, even though $\lim_{x \rightarrow 0} x \cdot \sin(\frac{1}{x}) = 0$!

Question for students

What does this mean about the complex-valued $\sin(\frac{1}{z})$ which has an isolated singularity at $z=0$?

An isolated singularity z_0 of f is called an essential singularity if it is not non-essential. By Riemann's removability criterion, $f(z)$ must be unbounded near z_0 if z_0 is an essential singularity, although this is also true for poles. In the case of poles, though, $|f(z)| \rightarrow \infty$ as $z \rightarrow z_0$. For essential singularities, not even this is true. (E.g., $\sin(\frac{1}{z})$ has an essential singularity @ 0.)

We won't prove the following theorem, but it highlights how "poorly behaved" functions are near essential singularities.

Thm (Picard's Big Theorem)

If z_0 is an essential singularity of

$f: U \setminus \{z_0\} \rightarrow \mathbb{C}$, then for every $r > 0$ such that $\bar{B}_r(z_0) \subseteq U$, $f(U) = \mathbb{C}$ or $f(U) = \mathbb{C} \setminus \{\xi\}$ for some ξ .

For most purposes, we don't like essential singularities because we don't have "nice" theorems for functions with essential singularities. As we'll see, though, poles aren't so problematic. So, we give functions whose only singularities are removable (which we often implicitly just "fill in" to remove the singularity) and poles a special name: they are called meromorphic functions.

It is sometimes convenient to imagine that ∞ is a point and a meromorphic function gives the value ∞ when evaluated at a pole. We make this precise using the "Riemann sphere."

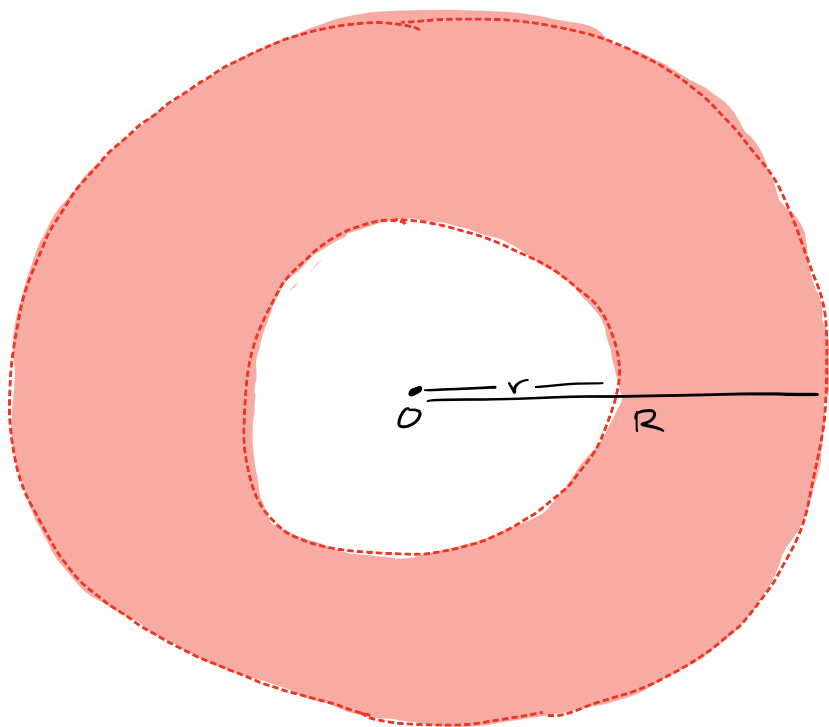
Math 424, Spring 2024
Lecture 22 - Laurent series

Meromorphic functions can always be represented as power series centered at points where the function is holomorphic, but can they be represented as series at their poles?

The function $f(z) = 1/z$ is an important example to consider. Can we write $1/z$ as a power series centered at 0? The answer is clearly "no" for our usual power series, since the coefficients would necessarily be (multiple of) derivatives of $1/z$ evaluated at 0, but the function is not defined there. If we're willing to extend our notion of a power series a little bit, though, it will turn out there is a way to represent meromorphic functions, such as $1/z$, by series centered at their poles.

First we need a definition: We define an annulus of inner radius r and outer radius R centered at 0 to be the set

$$A = \{z \in \mathbb{C} \mid r < |z| < R\}$$



Now let's consider functions which are holomorphic in A . Of course, any g which is holomorphic in $B_R(0)$ is holomorphic in A , but so are functions of the form $z \mapsto h(1/z)$ where h is defined and holomorphic in $B_{1/r}(0)$: if $|z| > r$, then $1/|z| < 1/r$. Notice too that given holomorphic $g: B_R(0) \rightarrow \mathbb{C}$ and $h: B_{1/r}(0) \rightarrow \mathbb{C}$, the map

$$\begin{aligned} f: A &\rightarrow \mathbb{C} \\ z &\mapsto g(z) + h\left(\frac{1}{z}\right) \end{aligned}$$

is holomorphic in A . In fact, if we choose our h so that $h(0) = 0$, every holomorphic function f defined on A has a unique decomposition as $f(z) = g(z) + h(1/z)$ — this is known as the Laurent decomposition of f .

Theorem

Let A be an annulus around 0 of inner radius r , outer radius R . Every holomorphic $f: A \rightarrow \mathbb{C}$ can be written uniquely as

$$f(z) = g(z) + h\left(\frac{1}{z}\right)$$

where $g: B_R(0) \rightarrow \mathbb{C}$ and $h: B_{\frac{1}{r}}(0) \rightarrow \mathbb{C}$ are holomorphic, and $h(0) = 0$.

Pf

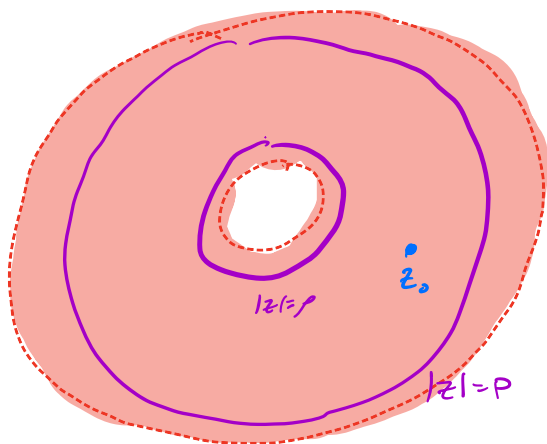
We must show this decomposition exists and is unique.

• Existence:

Let $z_0 \in A$ be fixed. Choose numbers ρ, P so that $0 < \rho < P < R$ and $\rho < |z_0| < P$.

Notice

$$\oint_{|z|=\rho} \frac{1}{z-z_0} dz = 0 \quad \text{and} \quad \oint_{|z|=P} \frac{1}{z-z_0} dz = 2\pi i$$



Now consider the function $G: A \rightarrow \mathbb{C}$ given by

$$G(z) = \begin{cases} \frac{f(z) - f(z_0)}{z - z_0} & \text{if } z \neq z_0 \\ f'(z_0) & \text{if } z = z_0 \end{cases}$$

G is holomorphic in A , so by the deformation theorem,

$$\oint_{|z|=\rho} G(z) dz = \oint_{|z|=\rho} G(z) dz$$

But note

$$\begin{aligned} \oint_{|z|=\rho} G(z) dz &= \oint_{|z|=\rho} \frac{f(z) - f(z_0)}{z - z_0} dz \\ &= \oint_{|z|=\rho} \frac{f(z)}{z - z_0} dz - f(z_0) \oint_{|z|=\rho} \frac{1}{z - z_0} dz \end{aligned}$$

and similarly,

$$\oint_{|z|=\rho} G(z) dz = \oint_{|z|=\rho} \frac{f(z)}{z - z_0} dz - f(z_0) \oint_{|z|=\rho} \frac{1}{z - z_0} dz$$

Thus

$$\oint_{|z|=\rho} \frac{f(z)}{z-z_0} dz - f(z_0) \oint_{|z|=\rho} \frac{1}{z-z_0} dz = \oint_{|z|=\rho} \frac{f(z)}{z-z_0} dz - f(z_0) \oint_{|z|=\rho} \frac{1}{z-z_0} dz$$

$$\Rightarrow \oint_{|z|=\rho} \frac{f(z)}{z-z_0} dz = \oint_{|z|=\rho} \frac{f(z)}{z-z_0} dz - f(z_0) \cdot 2\pi i$$

$$\Rightarrow f(z_0) = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{f(z)}{z-z_0} dz - \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{f(z)}{z-z_0} dz \quad (\star)$$

We now define g at $z_0 \in B_\rho(0)$ as

$$g(z_0) = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{f(z)}{z-z_0} dz$$

and for $z_0 \in B_\rho(0)$ we set

$$h(z_0) = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{z_0 f(z)}{1-z \cdot z_0} dz \quad \checkmark$$

Note $h(0) = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{0 \cdot f(z)}{1-0 \cdot z} dz$
 $= \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{0}{1} dz$
 $= 0$

hence $|z_0| < \frac{1}{\rho}$, so $\frac{1}{|z_0|} > \rho$ and

$$\begin{aligned} h\left(\frac{1}{z_0}\right) &= \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{\frac{1}{z_0} \cdot f(z)}{1-z \cdot (\frac{1}{z_0})} dz \\ &= \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{\frac{1}{z_0}}{\frac{1}{z_0}} \cdot \frac{f(z)}{z_0 - z} dz \\ &= \frac{-1}{2\pi i} \oint_{|z|=\rho} \frac{f(z)}{z - z_0} dz \end{aligned}$$

Then by our equation $(*)$ above we have

$$f(z_0) = g(z_0) + h\left(\frac{1}{z_0}\right).$$

We'll leave uniqueness as an exercise. $\square$

Given a holomorphic $f: A \rightarrow \mathbb{C}$ with Laurent decomposition $g(z) + h(\frac{1}{z})$, we often call $h(\frac{1}{z})$ the principal part of f ; $g(z)$ is the analytic part or regular part of f .

Notice the principal part determines a holomorphic function $h(z)$ as so has some power series expansion,

$$h(z) = \sum_{n=1}^{\infty} b_n z^n$$

Replacing z by $1/z = z^{-1}$, this becomes

$$h\left(\frac{1}{z}\right) = \sum_{n=1}^{\infty} b_n z^{-n} = \sum_{n=-1}^{-\infty} b_{-n} z^n$$

Combining this with the power series for $g(z)$, say

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

which we may combine together to write

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

This is the Laurent series of f

$$c_n = \begin{cases} a_n & \text{if } n \geq 0 \\ b_{-n} & \text{if } n < 0 \end{cases}$$

For example, consider $\frac{\sin(z)}{z}$, which is given by

$$\begin{aligned} \frac{\sin(z)}{z^4} &= \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots}{z^4} \\ &= z^{-3} - \frac{1}{3!} z^{-1} + \frac{1}{5!} z^1 - \frac{1}{7!} z^3 + \dots \end{aligned}$$

The principal part is $h\left(\frac{1}{z}\right) = z^{-3} + \frac{1}{3!} z^{-1}$, and the regular part is $g(z) = \frac{1}{5!} z - \frac{1}{7!} z^3 + \frac{1}{9!} z^5 + \dots$

In the case of $e^{1/z}$ we have

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3!z^3} + \dots$$

here the singular part is $h\left(\frac{1}{z}\right) = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3!z^3} + \dots$ and the regular part is $g(z) = 1$

As another example, the function $\frac{z^3 + 3z^2 + iz}{z}$ has the decomposition

$$\underbrace{i + 3z + z^2}_{g(z)} + \underbrace{0}_{h(1/z)}$$

Notice in our three examples we had a pole, an essential singularity, and a removable singularity and our principal part in the Laurent decomposition had finitely-many non-zero terms, infinitely-many non-zero terms, or no non-zero terms. This is true in general. Before giving that theorem, let's generalize our Laurent series a little bit

If A is an annulus with center z_0 ,

$$A = \{z \in \mathbb{C} \mid r < |z - z_0| < R\}$$

then the Laurent series of a holomorphic $f: A \rightarrow \mathbb{C}$ centered at z_0 is the series

$$\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

where

$$c_n = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

where $r < \rho < R$.

When z_0 is an isolated singularity of f , f is holomorphic in some annulus centered at z_0 with inner radius 0, and the type of singularity is determined by the number of non-zero coefficients on the $z^{\text{negative power}}$ terms.

Thm

Suppose $f: U \setminus \{z_0\} \rightarrow \mathbb{C}$ is holomorphic with isolated singularity z_0 and has Laurent series

$$\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

Then z_0 is removable iff $c_n = 0$ for all $n < 0$;
 z_0 is a pole iff $c_n = 0$ for all but finitely-many $n < 0$; and z_0 is an essential singularity iff $c_n \neq 0$ for infinitely-many $n < 0$.

Pf

(Easy) exercise. □

Math 424, Spring 2024
Lecture 23 - The Residue Theorem

In the last lecture we saw that if z_0 is an isolated singularity of a function f which is holomorphic in some punctured neighborhood of z_0 — so $f: U \setminus \{z_0\} \rightarrow \mathbb{C}$ is holomorphic where $U \subseteq \mathbb{C}$ is an open set containing z_0 — then f can be expressed as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

where the coefficients c_n can be computed by Cauchy's integral formula,

$$c_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz$$

From this series representation and expression of coefficients as integrals we can prove an extremely important, and useful, theorem called "the residue theorem." Before we define what a residue is and state the theorem, let's consider the following calculation.

Suppose z_0 is an isolated singularity of

f , and C is a simple closed curve encircling z_0 and no other isolated singularities of f . Let the Laurent series of f around z_0 be $\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$. Now consider

$$\begin{aligned} \oint_C f(z) dz &= \oint_C \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n dz \\ &= \oint_C \left[\sum_{n=-\infty}^{-2} c_n (z-z_0)^n + c_{-1} (z-z_0)^{-1} + \sum_{n=0}^{\infty} c_n (z-z_0)^n \right] dz \\ &= \sum_{n=-\infty}^{-2} \oint_C c_n (z-z_0)^n dz + \oint_C \frac{c_{-1}}{z-z_0} dz + \sum_{n=0}^{\infty} \oint_C c_n (z-z_0)^n dz \end{aligned}$$

Observe that for $n \neq -1$, $\oint_C (z-z_0)^n dz = 0$.

(Easy exercise: By performing the substitution $u = z - z_0$, we may suppose $z_0 = 0$. Then by the deformation theorem we can assume C is the unit circle. Finally, parametrize the circle as $z(t) = e^{it}$ for $0 \leq t \leq 2\pi$, and perform a direct computation.)

Thus we have

$$\oint_C f(z) dz = \oint_C \frac{c_{-1}}{z-z_0} dz = c_{-1} \cdot 2\pi i$$

So, integrals around a single isolated singularity are just $2\pi i$ times this coefficient c_{-1} that appears in our Laurent series expansion. Given the special significance of this value, we call it the residue of f at z_0 and denote it $\text{Res}(f; z_0)$.

In general, if a function has multiple isolated singularities, each one contributes its residue to the value of an integral of the function around a closed curve. This is known as the residue theorem:

Thm (Residue Theorem)

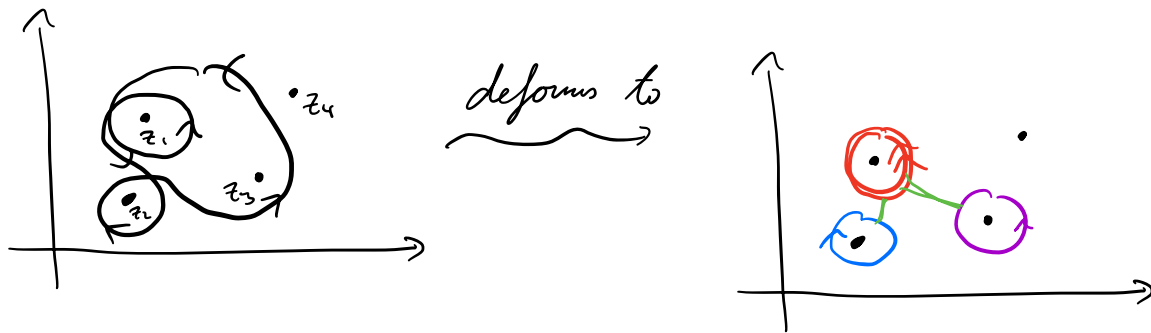
Suppose f is a holomorphic function with finitely-many isolated singularities $z_1, z_2, \dots, z_n$.

If C is any (piecewise smooth) closed curve in the domain of f (in particular, C does not pass through any z_k), then

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n n(C; z_k) \cdot \text{Res}(f; z_k)$$

Pf By the deformation theorem, we can suppose C decomposes into a concatenation of "multi-circles" $C_1, C_2, \dots, C_n$ where C_k encircles

z_k and no other isolated singularities of f , C_k consisting of $|n(C; z_k)|$ circles with positive orientation if $n(C; z_k) > 0$, negative orientation if $n(C; z_k) < 0$, and C_k is the "empty circle" if $n(C; z_k) = 0$.



We then apply our earlier calculation to each C_k circle, multiplied by $n(C; z_k)$ to account for orientation and C_k encircling z_k multiple times

□

So, the residue theorem essentially tells us evaluating integrals is the same as determining residues.

E.g., Compute $\oint_C e^{1/z} dz$ where C is the unit circle.

Notice $e^{1/z}$ has an essential singularity at 0. The Laurent series around 0 is

$$\sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n = 1 \cdot \frac{1}{z} + \frac{1}{2} \cdot \frac{1}{z^2} + \frac{1}{6} \cdot \frac{1}{z^3} + \dots$$

So $\text{Res}(e^{1/z}; 0) = 1$ and $\oint_C e^{1/z} dz = 2\pi i \cdot 1.$

If we already know the Laurent expansion of our function around its singularities, then we have the residues and can compute integrals easily. However, if we're given a function and asked to integrate it using the residue theorem, computing the Laurent decomposition doesn't seem helpful because that requires computing integrals!

However, we don't need the "full" Laurent decomposition, we just need this one coefficient. In some situations this can be computed more easily.

If z_0 is a removable singularity, then $\text{Res}(f; z_0) = 0.$

If z_0 is a pole of order m , then $(z - z_0)^m f(z)$ has a removable singularity and so can be expressed near z_0 as

$$(z - z_0)^m f(z)$$

$$= b_1 (z - z_0) + b_2 (z - z_0)^2 + \dots + b_{m-1} (z - z_0)^{m-1} + b_m (z - z_0)^m + b_{m+1} (z - z_0)^{m+1} + \dots$$

Notice b_{m-1} is the residue of f at z_0 as this becomes the coefficient of $\frac{1}{z-z_0}$ once we divide by z_0 . By Taylor's theorem (or the Cauchy integral formula, plus some minor arithmetic), this value, b_{m-1} , is $(m-1)!$ times the $(m-1)^{\text{th}}$ derivative of $(z-z_0)^m f(z)$ evaluated at z_0 .

(When $m=1$, this would be the zeroth derivative of $(z-z_0) \cdot f(z)$, which means evaluating the function at z_0 . This isn't quite defined, though, so we should interpret this as evaluating the function at its removable discontinuity, i.e., computing $\lim_{z \rightarrow z_0} (z-z_0) f(z)$.)

We can summarize this as follows:

Then

Suppose z_0 is a pole of $f(z)$. If z_0 is a simple pole, then

$$\text{Res}(f; z_0) = \lim_{z \rightarrow z_0} (z-z_0) \cdot f(z)$$

If z_0 is a pole of order $m > 1$, then

$$\text{Res}(f; z_0) = \frac{1}{(m-1)!} \left. \frac{d^{m-1}}{dz^{m-1}} \right|_{z=z_0} (z-z_0)^m f(z).$$

E.g.

- Find the residues of $f(z) = \frac{\sin(\pi z)}{z^3 - z^2}$ at each pole.

Note $f(z)$ has a pole of order 2 at 0 and a pole of order 1 at 1. Thus

$$\begin{aligned}\text{Res}(f; 0) &= \frac{1}{1!} \left. \frac{d}{dz} \right|_{z=0} z^2 \cdot \frac{\sin(\pi z)}{z^3 - z^2} \\&= \left. \frac{d}{dz} \right|_{z=0} \frac{\sin(\pi z)}{z-1} \\&= \left. \left(\frac{(z-1) \cdot \cos(\pi z) \pi - \sin(\pi z)}{(z-1)^2} \right) \right|_{z=0} \\&= \left(\frac{(-1) \cdot 1 \cdot \pi - 0}{1^2} \right) \\&= -\pi\end{aligned}$$

$$\text{Res}(f; 1) = \lim_{z \rightarrow 1} (z-1) \cdot \frac{\sin(\pi z)}{z^3 - z^2} = \lim_{z \rightarrow 1} \frac{\sin(\pi z)}{z^2} = \sin(\pi) = 0$$

So, for any simple closed curve C encircling both 0 and 1,

$$\begin{aligned}\oint_C \frac{\sin(\pi z)}{z^3 - z^2} dz &= 2\pi i \cdot \left(-\frac{\pi}{2} + 0 \right) \\&= -\pi^2 i\end{aligned}$$

As a special case of our theorem above, consider the case when our function has the form

$$f(z) = \frac{p(z)}{q(z)}$$

and has a simple pole at z_0 . In this case we may write

$$q(z) = (z - z_0) \cdot r(z)$$

where $r(z_0) \neq 0$. Notice the product rule gives us

$$q'(z) = r(z) + (z - z_0) \cdot r'(z)$$

But then $q'(z_0) = r(z_0)$, so

$$\begin{aligned} \lim_{z \rightarrow z_0} (z - z_0) f(z) &= \lim_{z \rightarrow z_0} (z - z_0) \cdot \frac{p(z)}{q(z)} \\ &= \lim_{z \rightarrow z_0} (z - z_0) \frac{p(z)}{(z - z_0) r(z)} \\ &= \lim_{z \rightarrow z_0} \frac{p(z)}{r(z)} \\ &= \frac{p(z_0)}{r(z_0)} \\ &= \frac{p(z_0)}{q'(z_0)} \end{aligned}$$

This gives us the following:

Then
 If $f(z) = \frac{p(z)}{q(z)}$ has a simple pole at z_0
 and $p(z_0)$ is defined, then

$$\text{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}$$

□

Example

Find the residues of all poles of

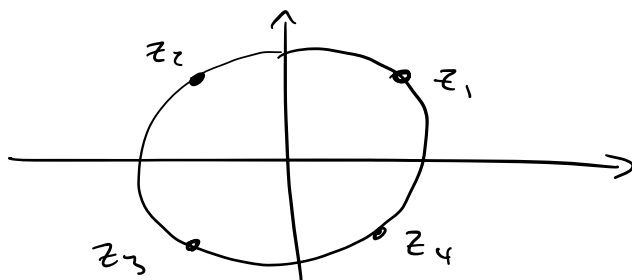
$$f(z) = \frac{z^2}{1+z^4}$$

Note the poles occur when $z^4 + 1 = 0$, or $z^4 = -1$. Writing $-1 = e^{\pi i}$, we see one such solution is $e^{i\pi/4}$. The other three roots occur at $e^{i[\pi/4 + m\pi/2]}$ for $m=1, 2, 3$ giving us

$$e^{i\pi/4}, \quad e^{i(\frac{\pi}{4} + \frac{\pi}{2})}, \quad e^{i(\frac{\pi}{4} + \pi)}, \quad e^{i(\frac{\pi}{4} + 6\pi/4)}$$

$$= e^{i3\pi/4}, \quad = e^{i5\pi/4}, \quad , \quad e^{i7\pi/4}$$

Let's call these z_1, z_2, z_3 , and z_4 .



Now note $\frac{d}{dz}(z^4+1) = 4z^3$, so

$$\text{Res}(f; z_1) = \frac{z_1^2}{4z_1^3} = \frac{1}{4z_1} = \frac{1}{4} e^{-i\pi/4}$$

$$\text{Res}(f; z_2) = \frac{z_2^2}{4z_2^3} = \frac{1}{4z_2} = \frac{1}{4} e^{-i3\pi/4}$$

$$\text{Res}(f; z_3) = \frac{1}{4} e^{-i5\pi/4}$$

$$\text{Res}(f; z_4) = \frac{1}{4} e^{-i7\pi/4}$$

We can sometimes use our theorems for complex integrals to evaluate otherwise difficult real integrals. E.g.,

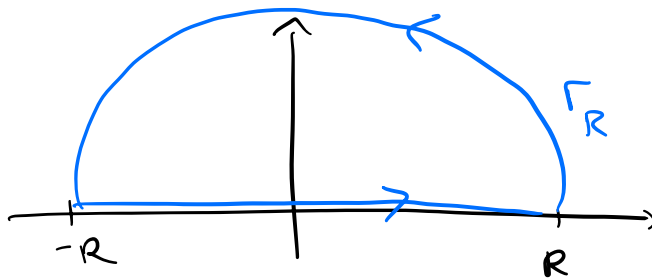
Compute $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx$. Recall this integral really refers to the limit

$$\lim_{R \rightarrow \infty} \int_{-R}^R \frac{x^2}{1+x^4} dx$$

Let's try to first compute $\int_{-R}^R \frac{x^2}{1+x^4} dx$

by thinking of this as a portion of the complex integral

$$\int_{\Gamma_R} \frac{z^2}{1+z^4} dz$$

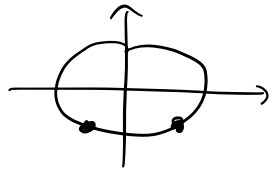


By the residue theorem, for $R > 1$ we have

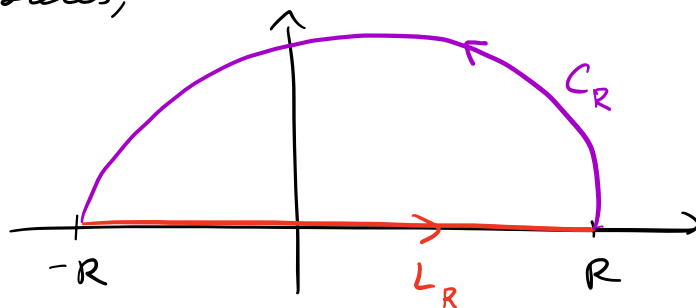
$$\int_{\Gamma_R} \frac{z^2}{1+z^4} dz = 2\pi i (\text{Res}(f; z_1) + \text{Res}(f; z_2))$$

where $f = \frac{z^2}{1+z^4}$ and z_1, z_2 are the poles from earlier, so we have

$$\begin{aligned} \int_{\Gamma_R} \frac{z^2}{1+z^4} dz &= 2\pi i \left(\frac{1}{4} e^{-i\pi/4} + \frac{1}{4} e^{-i3\pi/4} \right) \\ &= \frac{i\pi}{2} (e^{-i\pi/4} + e^{-i3\pi/4}) \\ &= \frac{i\pi}{2} \left(\frac{\sqrt{2}}{2} + \frac{-i\sqrt{2}}{2} + \frac{-\sqrt{2}}{2} + \frac{-i\sqrt{2}}{2} \right) \\ &= \frac{i\pi}{2} - (-i\sqrt{2}) \\ &= \frac{\pi}{\sqrt{2}} \end{aligned}$$



But we can also break this line integral into two pieces,



$$\begin{aligned}
 \int_{\Gamma_R} \frac{z^2}{1+z^4} dz &= \int_{L_R} \frac{z^2}{1+z^4} dz + \int_{C_R} \frac{z^2}{1+z^4} dz \\
 \Rightarrow \frac{\pi}{\sqrt{2}} &= \int_{-R}^R \frac{x^2}{1+x^4} dx + \int_{C_R} \frac{z^2}{1+z^4} dz \\
 \Rightarrow \int_{-R}^R \frac{x^2}{1+x^4} dx &= \frac{\pi}{\sqrt{2}} + \int_{C_R} \frac{z^2}{1+z^4} dz
 \end{aligned}$$

We may parametrize C_R by $z(t) = Re^{it}$, $0 \leq t \leq \pi$

$$\begin{aligned}
 \int_{C_R} \frac{z^2}{1+z^4} dz &= \int_0^\pi \frac{R^2 e^{i2t}}{1+R^4 e^{i4t}} \cdot Rie^{it} dt \\
 &= iR^3 \int_0^\pi \frac{e^{i2t}}{1+R^4 e^{i4t}} dt
 \end{aligned}$$

Notice

$$\begin{aligned}
 \left| iR^3 \int_0^\pi \frac{e^{i2t}}{1+R^4 e^{i4t}} dt \right| &\leq R^3 \int_0^\pi \left| \frac{e^{i2t}}{1+R^4 e^{i4t}} \right| dt \\
 &= R^3 \int_0^\pi \frac{1}{|1+R^4 e^{i4t}|} dt \\
 &\leq R^3 \int_0^\pi \frac{1}{1+R^4} dt
 \end{aligned}$$

$$= \frac{\pi R^3}{1 + R^4}$$

Notice as $R \rightarrow \infty$, this goes to zero,
and so

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx = \frac{\pi}{\sqrt{2}}$$