

Lecture 01 - Real Numbers

Real analysis is basically the study of "nice" functions of a real number. Here "nice" means continuous, differentiable, continuously differentiable, twice continuously differentiable, etc. A big part of real analysis is rigorously defining these "nice" properties and studying their consequences. That is, the starting point of real analysis is trying to put calculus on firm ground.

Calculus as developed by Newton and Leibniz was a bit hand-wavy. For example they talked about "infinitesimals" without exactly explaining what these were. In the 1800's, though, there was a push to make mathematics more formal, as it was discovered that being hand-wavy can lead to some surprises.
(E.g., the discovery of hyperbolic geometry by Gauss, Bolyai, and Lobachevsky.)

Before we can do that "formal calculus," though, we need to define exactly what a "real number" is. Let's begin by working our way up from simple sets.

Recall that the natural numbers are simply

the counting numbers $1, 2, 3, 4, 5, \dots$ and the set of all natural numbers is denoted $\mathbb{N}$.

We can add natural numbers together to obtain natural numbers. i.e., if $x, y \in \mathbb{N}$, then $x + y \in \mathbb{N}$. From this we may be interested in solving basic equations such as

$$x + 3 = 5$$

These equations may not always have natural number solutions, such as

$$x + 3 = 1$$

We can introduce new numbers to ensure these equations always have solutions and we arrive at the integers, $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

If we consider multiplication of integers, we again see that not all equations will have solutions. E.g., there is no integer solution to the equation

$$2x = 1$$

Solving these equations require us to introduce the rational numbers. The set of rational numbers, denoted $\mathbb{Q}$, consists of ratios of integers with non-zero numerator:

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\} / \sim$$

the " $/ \sim$ " means we're really considering equivalence classes. E.g., we want to consider $\frac{3}{2} = \frac{6}{4}$. The equivalence relation is

$$\frac{a}{b} \sim \frac{c}{d} \text{ iff } ad = bc.$$

Before jumping to our next set of numbers, the reals, we should discuss some properties of the rational numbers.

Lemma

Between any two distinct rational numbers there exists a rational number between them.

Pf

Suppose our numbers are $p = \frac{a}{b}$ and $q = \frac{c}{d}$ with $p < q$. Note we may rewrite our numbers as

$$p = \frac{ad}{bd} \quad \text{and} \quad q = \frac{bc}{bd}$$

As $p < q$, we must have $ad < bc$. ★ HW

Let

$$r = \frac{p+q}{2} = \frac{ad+bc}{2bd} = \frac{\left(\frac{ad+bc}{2}\right)}{bd}$$

As $ad < \frac{ad+bc}{2} < bc$, ★ HW $p < r < q$, and r is rational as it can be written as a ratio of integers ($ad+bc$ over $2bd$) with non-zero denominator. □

Cor

Between any two distinct rational numbers there are infinitely-many rational numbers

Pf

★ HW

Another important property of the rationals is that they form an "ordered field," which we'll define bit-by-bit.

A field is a non-empty set F together with two binary operators $+$ and $\cdot$.

(i.e., maps $+: F \times F \rightarrow F$ and $\cdot: F \times F \rightarrow F$ where

we usually write $a+b$ for $+(a,b)$ and $a \cdot b$ or simply ab for $\cdot(a,b)$) satisfying the following condition for all $a, b, c \in F$

- $a + b = b + a$
- $a + (b + c) = (a + b) + c$
- There exists an element $0 \in F$ such that $a + 0 = a$
- There exists an element $-a \in F$ such that $a + (-a) = 0$
- $ab = ba$
- $a(bc) = (ab)c$
- There exists an element $1 \in F$ such that $1 \cdot a = a$
- If $a \neq 0$, there exists an element a^{-1} such that $a \cdot a^{-1} = 1$
- $a(b + c) = ab + ac$

Basically a "field" is just a set where we can do arithmetic and all the usual properties

we'd expect will hold.

It's tedious to do so, but we can check that $\mathbb{Q}$ is a field.

In addition to being a field, $\mathbb{Q}$ also has an ordering of its elements. That is, there exists a relation $<$ on $\mathbb{Q}$ such that for every pair of elements $a, b \in \mathbb{Q}$, exactly one of the following statements is true:

$$a < b, \quad b < a, \quad \text{or} \quad a = b.$$

and additionally this relation is transitive:

$$\text{if } a < b \text{ and } b < c, \text{ then } a < c.$$

These two "structures" on $\mathbb{Q}$ — the field structure and the order relation — are not independent of each other. That is $\mathbb{Q}$ is an example of an ordered field, meaning the following:

- if $a, b > 0$, then $a + b > 0$ and $ab > 0$; and
- $a > b$ iff $a - b > 0$.

It's pretty easy to see that every field can be given an order, but not all orders on a field will be "compatible" with the field operation.

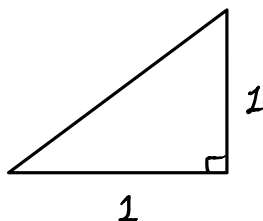
For example, it's easy (but tedious) to check $\{0, 1\}$ is a field if we define

$$0+0=0, \quad 0+1=1, \quad 1+0=1, \quad 1+1=0$$

$$0 \cdot 0=0, \quad 0 \cdot 1=0, \quad 1 \cdot 0=0, \quad 1 \cdot 1=1$$

We can also put an order on $\{0, 1\}$ by declaring $0 < 1$. This is not an ordered field, though, since for example, $1 > 0, 1 > 0$, but $1+1=0 \not> 0$.

Though the rational numbers have several nice properties, they are not sufficient for all purposes. It was recognized in antiquity, for example, that some naturally occurring values, such as lengths, may not be rational. For example, consider the hypotenuse of an isosceles right triangle where the non-hypotenuse sides have length 1.



By the Pythagorean theorem, the hypotenuse of this triangle has length $\sqrt{2}$. We claim, though, that $\sqrt{2}$ is not rational.

Thm

$\sqrt{2}$ is not rational — it can not be written as a ratio of two integers.

Pf

Suppose for the sake of contradiction that $\sqrt{2}$ was rational: say $\sqrt{2} = \frac{p}{q}$ for integers p and q , $q \neq 0$. By cancelling any common factors, we may suppose $\frac{p}{q}$ is in lowest terms. Now note

$$2 = \sqrt{2}^2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2} \\ \Rightarrow 2q^2 = p^2$$

Thus p^2 is even, and so p must be even as well (prove this in HW). In fact, since 2 divides p , we must have that 4 divides p^2 . But $q^2 = \frac{p^2}{2}$, so q^2 must also be even, and so q is even. This contradicts our assertion that p and q are in lowest terms, hence $\sqrt{2}$ is irrational. □

Thus the rationals are "missing" some numbers we care about, and we need to add these "missing" numbers back in. When we add these missing numbers, we arrive at the real numbers.

Defining a real number carefully is a bit technical, but we'll see one way to do it soon. For now, though, we'll define the reals non-constructively by describing them as a set with some special properties. For the purposes of proving theorems about real numbers this is usually good enough — any proof that only uses those assumed properties will be valid. We'll just have to take it on faith that a set with our desired properties actually exists, and we'll show this set exists by constructing the real numbers as certain equivalence classes of sequences of rational numbers later.

Lecture 02 - Distance and absolute value properties

In the last lecture we said that we would treat the set of real numbers as a set with some special properties, and later construct real numbers as certain sequences of rational numbers. To do this, it will be helpful to recall some basic facts about distance between numbers.

Since we haven't yet defined the real numbers, we shouldn't really be stating theorems involving real numbers and should really only prove results for rational numbers. Many of the theorems and proofs we'll give right now will apply equally well for real numbers, even if *a priori* we can only use rational numbers for the time being. Because of this we will use the slightly ambiguous word "number" here which for now really means a rational number, but which (after we've defined them) can mean real number later.

First, a definition: given two numbers x and y , we define the distance between

x and y to be $|x-y|$.

E.g., The distance between 3 and $\frac{19}{4}$ is

$$\left| 3 - \frac{19}{4} \right| = \left| \frac{12 - 19}{4} \right| = \left| -\frac{7}{4} \right| = \frac{7}{4}$$

Since our notion of distance is defined in terms of absolute values, it will be convenient to collect a few basic properties of absolute values.

Let's first recall that the absolute value is defined as a piecewise function,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

From this definition we can prove several useful properties of the absolute value:

Thm

- ° For any number x and any number $a > 0$,
 $|x| < a$ iff $-a < x < a$
- ° For any pair of numbers x and y , $|xy| = |x||y|$
- ° For any number x , $|x| = |-x|$.
- ° For any number x , $-|x| \leq x \leq |x|$.

Pf

We'll only prove the first part of the theorem and leave the rest as an exercise.

Let x be any number and a any number with $a > 0$.

Suppose first that $|x| < a$. If $x \geq 0$, then this means $|x| = x$, so $x < a$.

Now, as $a > 0$ we have $-a < 0$. As $x \geq 0$, though, and since the relation is transitive,
$$-a < x < a.$$

If $x < 0$, then $|x| = -x$, so $-x < a$, and so $x > -a$. Since $x < 0$, $x < -x$ and we have

$$\begin{aligned} -a &< x < -x < a \\ \Rightarrow -a &< x < a. \end{aligned}$$

We now prove the converse, so suppose $-a < x < a$ where $a > 0$. If $x \geq 0$, then $|x| = x$ and $|x| < a$ follows from our assumption $x < a$. If $x < 0$, then $|x| = -x$. As since $-a < x$, $a > -x$, thus $|x| < a$. □

Another extremely useful property is the triangle inequality.

Thm (Triangle inequality)

For any numbers x and y ,

$$|x + y| \leq |x| + |y|$$

Pf

We know from the theorem above that

$$-|x| \leq x \leq |x| \quad \text{and} \quad -|y| \leq y \leq |y|.$$

Adding these inequalities together we have the following (that this can be done follows

from the fact we have an ordered field),

$$-|x| - |y| \leq x + y \leq |x| + |y|$$

The above theorem again tells us

$$|x + y| \leq |x| + |y|.$$

□

It is hard to over-state how useful the triangle inequality will be for us in this class. Often we'll want to compare distances and say a given distance isn't "too big" - i.e., we want a bound on how large a distance might be, and the triangle inequality is very useful here.

Another useful thing is the reverse triangle inequality

Thm (Reverse triangle inequality)

For all numbers x, y , we have

$$||x| - |y|| \leq |x - y|$$

Pf

If $x=y$, this becomes $0 \leq 0$, so suppose $x \neq y$. We may use the (normal) triangle inequality to write

$$\begin{aligned}|x| &= |x-y+y| \leq |x-y| + |y| \\ |y| &= |y-x+x| \leq |y-x| + |x|\end{aligned}$$

Moving the term on the right which involves only one variable over to the left, this becomes

$$\begin{aligned}|x| - |y| &\leq |x-y|, \text{ and} \\ |y| - |x| &\leq |y-x|\end{aligned}$$

The second inequality we may rewrite as follows:

$$\begin{aligned}|y| - |x| &\leq |y-x| \\ \Rightarrow |y| - |x| &\leq |x-y| \\ \Rightarrow -(|y| - |x|) &\geq -|x-y| \\ \Rightarrow -|x-y| &\leq |x| - |y|\end{aligned}$$

Combining this with our earlier inequality we have

$$-|x-y| \leq |x|-|y| \leq |x-y|$$

Thus

$$||x|-|y|| \leq |x-y|.$$

□

We will see both the triangle and reverse triangle inequalities used repeatedly in this course, so it is worthwhile to spend some time committing them to memory.

We will mention one more useful inequality:

Thm (Cauchy-Schwarz Inequality)

Given any collection of numbers $x_1, y_1, x_2, y_2, \dots, x_n, y_n$

$$\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right)$$

Furthermore, this inequality is upgraded to an equation iff there exists a λ such that $b_i = \lambda a_i$ for each i .

Pf
Notice

$$\begin{aligned} 0 &\leq \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 \\ &= \sum_{i=1}^n \sum_{j=1}^n [(a_i b_j)^2 - 2 a_i a_j b_i b_j + (a_j b_i)^2] \\ &= \sum_{i=1}^n \sum_{j=1}^n a_i^2 b_j^2 - 2 \sum_{i=1}^n \sum_{j=1}^n a_i a_j b_i b_j + \sum_{i=1}^n \sum_{j=1}^n a_j^2 b_i^2 \\ &= \sum_{i=1}^n a_i^2 \sum_{j=1}^n b_j^2 - 2 \sum_{i=1}^n a_i b_i \sum_{j=1}^n a_j b_j + \sum_{i=1}^n b_i^2 \sum_{j=1}^n a_j^2 \\ &= 2 \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - 2 \sum_{i=1}^n a_i b_i \sum_{i=1}^n a_i b_i \end{aligned}$$

The last step is simply factoring and rewriting j as i . We now have

$$\begin{aligned} 0 &\leq 2 \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right) - 2 \left(\sum_{i=1}^n a_i b_i \right)^2 \\ &\Rightarrow 2 \left(\sum_{i=1}^n a_i b_i \right)^2 \leq 2 \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right) \end{aligned}$$

Dividing both sides by 2 gives the Cauchy-Schwarz inequality. $\square$

Lecture 03- Decimal Expansions

As previously mentioned, one of the first goals of real analysis is to make calculus more formal. Since everything in calculus is defined in terms of limits, we'll care a lot about limits in this class, and here is where we see one particular failure of the rational numbers: the limit of a sequence of rational numbers need not be rational. To understand this, it will be convenient to use decimal expansions.

First note every rational number admits a decimal expansion, such as

$$\frac{1}{8} = 0.125$$

$$\frac{17}{13} = 1.307692307692307692\dots$$

As the second example shows, this decimal expansion need not be finite. However, adopting the convention that a finite decimal expansion is followed by infinitely-many zeros, we do have the following characterization of decimal expansions of rational numbers:

Theorem

Every rational number may be represented as an eventually repeating decimal expansion, and every eventually repeating decimal expansion represents a rational number. □

Before proving this theorem, let's be sure we understand what "eventually repeating" means. For this, let's write a decimal expansion as

$$D_m D_{m-1} \dots D_2 D_1 D_0 . d_1 d_2 d_3 \dots$$

where each D_i and d_i is a digit:

$$D_i, d_i \in \{0, 1, 2, \dots, 9\}$$

We'll say this decimal expansion is eventually repeating if there exists $N, p \in \mathbb{N}$ such that

$$d_{N+i} = d_{N+i+p}$$

for every $i \in \mathbb{N}$.

For example, consider the expansion

123.45678678678678...

Here $N=2$ and $p=3$ since

$$d_3 = d_6 = d_9 = d_{12} = \dots = 6$$

$$d_4 = d_7 = d_{10} = d_{13} = \dots = 7$$

$$d_5 = d_8 = d_{11} = d_{14} = \dots = 8$$

Thus the sequence of digits 678, which has length $p=3$, continually repeats once we're past the second ($N=2$) digit after the decimal point

123.45 $\overbrace{678}^{p=3}$ 678678...
 $\underbrace{\hspace{1.5cm}}_{N=2}$

And notice again we treat finite decimal expansions as eventually repeating expansions where 0 repeats. E.g.,

0.125 = 0.125 $\overbrace{0000}^{p=1}$...
 $\underbrace{\hspace{1.5cm}}_{N=3}$

With our definition of "eventually repeating" out of the way, we can start proving our theorem above. We will prove this in two halves; the first half will be the easier part of the theorem, which is actually the second statement above (eventually repeating $\Rightarrow$ rational). After proving that statement, we'll need to take a small detour to recall some simple facts about arithmetic of integers to prepare for the proof of the other half of the theorem.

Pf that eventually repeating decimals are rational

Suppose x is an eventually repeating decimal,

$$x = D_m \dots D_2 D_1 D_0 . d_1 d_2 \dots d_N d_{N+1} \dots d_{N+p} d_{N+p+1} \dots$$

where we use the convention above that

$$d_{N+i} = d_{N+i+p}. \quad \text{Notice}$$

$$10^N x = D_m \dots D_1 D_0 d_1 d_2 \dots d_N . d_{N+1} \dots d_{N+p} d_{N+p+1} \dots$$

Now let y be the repeating decimal to the right of the decimal in the expression above:

$$y = 0.d_{N+1} \dots d_{N+p} d_{N+p+1} \dots$$

and let $z = \lfloor 10^N x \rfloor$, so

$$z = D_m \dots D_1 D_0 d_1 \dots d_N$$

Notice $10^N x = z + y$, so $x = \frac{z+y}{10^N}$. As the rational numbers form a field, it now suffices to show that y is rational, since z and 10^N are integers and hence rational. This is accomplished by some simple arithmetic and algebra:

$$y = 0.d_{N+1} \dots d_{N+p} d_{N+p+1} \dots$$

$$\Rightarrow 10^p y = d_{N+1} \dots d_{N+p} \cdot d_{N+p+1} \dots$$

$$\Rightarrow 10^p y - y = d_{N+1} \dots d_{N+p}$$

The last step follows from the fact $d_{N+i} = d_{N+i+p}$ for each $i \in \mathbb{N}$. Notice $10^p y - y$ is an integer. Now we simply solve for y to obtain

$$\begin{aligned} y(10^p - 1) &= d_{N+1} \dots d_{N+p} \\ \Rightarrow y &= \frac{d_{N+1} \dots d_{N+p}}{10^p - 1} \end{aligned}$$

As this is a ratio of integers, the proof is complete. $\square$

To understand the proof more thoroughly, it may be helpful to see how our earlier eventually repeating decimal, $123.45678678\dots$, can be written as a ratio of integers by applying the ideas of the proof.

We set $x = 123.45678678\dots$. Recall $N=2$, $p=3$ for this decimal. Now notice

$$10^2 x = 12345.678678678\dots$$

We then have $y = 0.678678678\dots$, $z = 12345$, so

$$\begin{aligned} 10^2 x &= 12345 + 0.678678678\dots \\ \Rightarrow x &= \frac{12345 + 0.678678678\dots}{100} \end{aligned}$$

Now we turn our attention to writing $0.678678678\dots$ as a ratio of integers, which we do as follows:

$$\begin{aligned}
 10^3 y &= 678.678678678\ldots \\
 \Rightarrow 10^3 y - y &= 678.678678678\ldots - 0.678678678\ldots \\
 &= 678 \\
 \Rightarrow y(10^3 - 1) &= 678 \\
 \Rightarrow y &= \frac{678}{999}
 \end{aligned}$$

Plugging this into our earlier expression for x gives us

$$\begin{aligned}
 123.45678678678\ldots = x &= \frac{12345 + 0.678678678\ldots}{100} \\
 &= \frac{12345 + \frac{678}{999}}{100} \\
 &= \frac{12345 \cdot \frac{999}{999} + \frac{678}{999}}{100} \\
 &= \frac{\left(\frac{12332655 + 678}{999} \right)}{100} \\
 &= \frac{12333333}{99900}
 \end{aligned}$$

To show the other part of the proof (every rational number may be written as an eventually repeating decimal), it will be helpful to recall the division algorithm of Euclid:

Thm (The division algorithm)

For every pair of integers p and q with $q > 0$, there exists a unique pair of integers m and r with $0 \leq r < q$, such that

$$p = mq + r$$

Pf

Let m be the largest integer such that $mq \leq p$. Now define r to be $p - mq$.

Since $p \geq mq$, $r = p - mq$ is non-negative. Now notice

$$mq + r = mq + p - mq = p.$$

We still need to show $r < q$.

If r were greater than or equal to q , then we could write $r = q + s$ for some $s \geq 0$, but then

$$p = mq + r = mq + q + s = (m+1)q + s$$

As $s \geq 0$, though, $(m+1)q = p - s \leq p$, and this contradicts the fact m is the largest integer such that $mq \leq p$.

We still need to show m and r are the unique pair of integers satisfying $0 \leq r < q$ and $p = mq + r$. Suppose there was another pair of integers m', r' such that $p = m'q + r'$ with $0 \leq r' < q$. We'd then have

$$p = mq + r = m'q + r'$$

$$\Rightarrow mq - m'q + r = r'$$

$$\Rightarrow (m - m')q + r = r'$$

As r and r' are both between 0 and q (not including q), we must have $m - m' = 0$.

If instead $m - m' \geq 1$, then

$$\begin{aligned} r' &= (m - m')q + r \\ &\geq 1 \cdot q + r \\ &\geq q \end{aligned}$$

which contradicts $0 \leq r' < q$. Similarly, if $m - m' \leq -1$, then

$$r' = (m - m')q + r \leq -q + r < 0$$

As $m - m' = 0$, $m = m'$, so we have

$$\begin{aligned}mq + r &= m'q + r' \\ \Rightarrow mq + r &= mq + r' \\ \Rightarrow r &= r'\end{aligned}$$

□

We are now ready to prove the other part of our theorem above.

Pf every rational number can be written as an eventually repeating decimal

Consider the rational number $\frac{p}{q}$. By the division algorithm, there exist integers m_0, r_0 with $0 \leq r_0 < q$ such that

$$p = m_0 q + r_0$$

Dividing both sides by q this becomes

$$\frac{p}{q} = m_0 + \frac{r_0}{q}$$

← Note $0 \leq r_0 < q$, so $0 \leq \frac{r_0}{q} < 1$

If $r_0 = 0$, we are done: $\frac{p}{q} = m_0 = m_0.000\dots$

Otherwise let n_1 be the smallest integer so $10^{n_1} r_0 \geq q$.

We may write $\frac{r_0}{q}$ as $\frac{1}{10^{n_1}} \cdot \frac{10^{n_1} r_0}{q}$, so

$$\frac{p}{q} = m_0 + \frac{1}{10^{n_1}} \cdot \frac{10^{n_1} r_0}{q}$$

Applying the division algorithm again, there are integers m_1, r_1 with $0 \leq r_1 < q$ so that

$$10^{n_1} r_0 = m_1 q + r_1$$

Hence

$$\frac{10^{n_1} r_0}{q} = m_1 + \frac{r_1}{q}$$

Thus

$$p = m_0 + \frac{1}{10^{n_1}} \left(m_1 + \frac{r_1}{q} \right)$$

If $r_1 = 0$, we are done:

$$p = m_0 \cdot \underbrace{0 \dots 0}_{n_1-1 \text{ zeros}} m_1$$

Otherwise, let n_2 be the smallest integer with $10^{n_2} r_1 \geq q$.

We may then write

$$p = m_0 + \frac{1}{10^{n_1}} \left(m_1 + \frac{1}{10^{n_2}} \cdot \frac{10^{n_2} r_1}{q} \right)$$

Applying the division algorithm to $10^{n_2} r_1$ and q , we find integers m_2, r_2 with $0 \leq r_2 < q$ and

$$\begin{aligned} 10^{n_2} r_1 &= m_2 q + r_2 \\ \Rightarrow \frac{10^{n_2} r_1}{q} &= m_2 + \frac{r_2}{q} \end{aligned}$$

Hence

$$\frac{p}{q} = m_0 + \frac{1}{10^{n_1}} \left(m_1 + \frac{1}{10^{n_2}} \left(m_2 + \frac{r_2}{q} \right) \right)$$

Continuing in this way we eventually reach a point where $r_i = 0$ and have

$$\frac{p}{q} = m_0 + \frac{1}{10^{n_1}} \left(m_1 + \frac{1}{10^{n_2}} \left(m_2 + \dots + \frac{1}{10^{n_i}} m_i \right) \dots \right)$$

which gives us the finite decimal expansion

$$m_0. \underbrace{0 \dots 0}_{n_1-1} m_1 \underbrace{0 \dots 0}_{n_2-1} m_2 \dots m_i$$

Or, since each remainder r_i takes on one of

finitely many values (as $0 \leq r_i < q$), an r_i repeats. Once one r_i repeats, the sequence of m_i 's and n_i 's repeats, and we write

$$\frac{p}{q} = m_0 + \frac{1}{10^{n_1}} (m_1 + \frac{1}{10^{n_2}} (m_2 + \frac{1}{10^{n_3}} (m_3 + \dots$$

which gives us the decimal expansion

$$m_0 \cdot \underbrace{0 \dots 0}_{n_1-1} m_1 \underbrace{0 \dots 0}_{n_2-1} m_2 \underbrace{0 \dots 0}_{n_3-1} \dots \underbrace{0 \dots 0}_{n_i-1} m_i \underbrace{0 \dots 0}_{n_{i+1}-1} \dots$$

□

To make the proof above a bit more concrete, let's apply the ideas of the proof to find the decimal expansion of $\frac{31}{22}$.

We first note

$$p=31, q=22$$

$$\begin{aligned} 31 &= 1 \cdot 22 + 9 \\ \Rightarrow \frac{31}{22} &= 1 + \frac{9}{22} \end{aligned}$$

$$m_0=1, r_0=9$$

Notice $10^1 \cdot 9 = 90 > 22$, and so we have

$$n_1=1$$

$$\frac{31}{22} = 1 + \frac{1}{10} \cdot \frac{90}{22}$$

Now we use the division algorithm to write

$$\begin{aligned} 90 &= 4 \cdot 22 + 2 \\ \Rightarrow \frac{90}{22} &= 4 + \frac{2}{22} \end{aligned} \quad m_1 = 4, \quad r_1 = 2$$

Hence

$$\frac{31}{22} = 1 + \frac{1}{10} \left(4 + \frac{2}{22} \right)$$

Now notice $10^2 \cdot 2 = 200 > 22$, so $m_2 = 2$

$$\frac{31}{22} = 1 + \frac{1}{10} \left(4 + \frac{1}{100} \cdot \frac{200}{22} \right)$$

Using the division algorithm

$$\begin{aligned} 200 &= 9 \cdot 22 + 2 \\ \Rightarrow \frac{200}{22} &= 9 + \frac{2}{22} \end{aligned} \quad m_3 = 9, \quad r_3 = 2$$

We may write

$$\frac{31}{22} = 1 + \frac{1}{10} \left(4 + \frac{1}{100} \left(9 + \frac{2}{22} \right) \right)$$

Notice our remainder has already repeated: $r_2 = r_3 = 2$.
The sequence of m_i 's and r_i 's will repeat

from now on. For example, the next step of our process give us:

$$10^2 \cdot 2 = 200 > 22$$

$$200 = 9 \cdot 22 + 2$$

$$n_3 = 2$$

$$m_4 = 9, r_4 = 2$$

So,

$$\frac{31}{22} = 1 + \frac{1}{10} \left(4 + \frac{1}{100} \left(9 + \frac{1}{100} \left(9 + \frac{2}{22} \right) \right) \right)$$

Continuing the process gives

$$\frac{31}{22} = 1 + \frac{1}{10} \left(4 + \frac{1}{100} \left(9 + \frac{1}{100} \left(9 + \frac{1}{100} \left(9 + \frac{1}{100} \left(9 + \dots \right) \right) \right) \right) \right)$$

Which we may write as

$$\begin{aligned} \frac{31}{22} &= 1 + \frac{4}{10} + \frac{9}{1000} + \frac{9}{100000} + \frac{9}{10000000} + \dots \\ &= 1 + 0.4 + 0.009 + 0.00009 + 0.0000009 + \dots \\ &= 1.40909090909\dots \end{aligned}$$

It's worth pointing out here that the process described above is essentially the same as long division. The step in long division where you

add additional zeros to number under the division symbol is tantamount to our step of multiplying a remainder r_i by $10^{n_{fix}}$:

$$\begin{array}{r}
 1.40909\ldots \\
 22 \overline{) 31} \text{ } 0000 \text{ } \text{The purple zeros are our } \frac{2}{22} = \frac{1}{10} \cdot \frac{200}{22} \\
 \underline{-22} \text{ } \uparrow \text{ adding these zeros is like our } \\
 90 \text{ } \downarrow \frac{9}{22} = \frac{1}{10} \cdot \frac{90}{22} \text{ step above} \\
 \underline{-88} \\
 200 \\
 \underline{-198} \\
 200 \\
 \underline{-198} \\
 2 \\
 \vdots
 \end{array}$$

Lecture 04 - Completeness

In the last lecture we saw that every rational number has an eventually repeating decimal expansion and every eventually repeating decimal expansion represents a rational number. The contrapositives of these statements thus show the following:

Cor

Every irrational number is represented by a decimal expansion which is not eventually repeating, and every decimal expansion which is not eventually repeating represents an irrational number.

By the corollary, the following decimal expansion represents an irrational number:

$0.1234567891011121314\dots$

Notice that we can (at least, intuitively, we should be able to) construct a sequence of rational numbers that approaches this irrational number:

$$x_1 = 0.1, \quad x_2 = 0.12, \quad x_3 = 0.123, \dots \quad x_{12} = 0.123456789101112, \dots$$

That is, it seems that we can have a sequence of rational numbers whose limit is not rational. It is for this reason that rational numbers will be inadequate for the purposes of calculus as everything in calculus is defined in terms of limits.

This means we will want real numbers to be a set with the important property that the limit of a sequence of real numbers is a real number. This property can be rephrased in various equivalent ways which are more appropriate for us right now since we haven't actually defined a limit yet. These other formulations of our desired property will require us to set up some preliminary vocabulary, however.

Keep in mind that we are using the slightly ambiguous phrase "number" to mean a rational number for now, but statements about "numbers" will also hold for real

numbers, once we define what a real number is.

We say a set of numbers S is bounded above if there exists a number M such that $s \leq M$ for every $s \in S$. We will refer to this number M as an upper bound for S . For example, the set

$$S = \{0.9, 0.99, 0.999, \dots\}$$

is bounded above with upper bound $M=1$.

Notice that upper bounds are not unique: if M is an upper bound for S and M' is any number with $M \leq M'$, then M' is also an upper bound. For example, $M'=2$ is also an upper bound for the set $S = \{0.9, 0.99, 0.999, \dots\}$ above.

If a set is not bounded above, then for every number N , there exists an $s \in S$ with $s > N$. For example, consider the set $S = \{2^n \mid n \in \mathbb{N}\} = \{2, 4, 8, 16, \dots\}$. Given any

number N , there exists an $s \in S$ such that $s > N$: if $N \leq 0$, then $N < 2$; if $N > 0$, then $N < \lceil N \rceil$ and $\lceil N \rceil < 2^{\lceil N \rceil}$, but $2^{\lceil N \rceil} \in S$.

When a set S is bounded above, we will often care about the smallest possible upper bound which is called the supremum of the set and denoted $\sup S$. That is, the supremum (if it exists) is a number M which is an upper bound for S and has the property that if M' is any other upper bound, then $M \leq M'$.

For example, we claim that 1 is the supremum of $S = \{0.9, 0.99, 0.999, \dots\}$. We already know 1 is an upper bound, but need to show that it's the smallest possible upper bound. Suppose, for the sake of contradiction, there was some smaller upper bound M . Then for every $s \in S$ we'd have $s \leq M < 1$. As $M < 1$, $1 - M > 0$, but we can find an $s \in S$ with $1 - s < 1 - M$. For example, say $1 - M < \frac{1}{10^n}$. Then take $s = 1 - \frac{1}{10^{n+1}} = 0.\underbrace{99\dots 9}_{n+1 \text{ 9's}}$. Notice $1 - s = 1 - (1 - \frac{1}{10^{n+1}}) = \frac{1}{10^{n+1}} < \frac{1}{10^n}$.

But then we have

$$1-s < 1-M$$

$$\Rightarrow -s < -M$$

$$\Rightarrow s > M$$

and this contradicts the statement that M is an upper bound of S .

Even though upper bounds of a set are not unique, suprema are (provided they exist)

Prop

If the supremum of a set S exists, it is unique

Pf

Homework.

"Upper bounded," "bounded above," and "supremum" are all defined in terms of inequalities - saying elements of S are no larger than something. By reversing the inequalities (saying elements of S are no smaller than something), we have

the following definitions

- A set of numbers S is bounded below if there exists a number m such that $m \leq s$ for every $s \in S$.
- A number m satisfying $m \leq s$ for every $s \in S$ is called a lower bound of S .
- The largest lower bound of a set S bounded below is called the infimum of S and denoted $\inf S$.

For example, $S = \{1/2, 1/3, 1/4, 1/5, \dots\}$ is bounded below and $\inf S = 0$.

If a set S is bounded above and bounded below, then we simply say S is bounded.

As some of our examples indicate, the supremum and infimum of a set S may or may not be an element of S .

A set need not have a supremum or infimum. For example a set which is not bounded above

does not have a supremum. A more interesting question is whether every set bounded above has a supremum and whether every set bounded below has an infimum. The answer to these questions depends on what a "number" is.

For example, consider the set of all rational numbers with the property that the number's square is less than 2:

$$S = \{s \in \mathbb{Q} \mid s^2 < 2\}$$

This set is bounded above: for example 2 is an upper bound: if $s \in S$, then $s^2 < 2$; we want to show $s < 2$. If $s \leq 1$, then $s < 2$ and we're done. So suppose $s > 1$. Then $s < s^2$ and since $s^2 < 2$, $s < s^2 < 2$, so $s < 2$.

So, S has an upper bound, but does it have a least upper bound? We claim there is no rational number $\frac{a}{b}$ which is a least upper bound. If we can verify this, that will mean having an upper bound is not

good enough to guarantee the existence of a supremum if we are limited to the rational numbers.

Let's first show there is no rational supremum to our set S above, though.

Suppose $\sup S = \frac{a}{b} \in \mathbb{Q}$. Then for every $s \in S$ we'd have $s \leq \frac{a}{b}$ and $s^2 < 2$. Now consider two cases: whether $\frac{a}{b} \in S$ or $\frac{a}{b} \notin S$.

If $\frac{a}{b} \in S$, then $\frac{a^2}{b^2} < 2$, so $0 < 2 - \frac{a^2}{b^2}$.

Now let n be any ^{pos.} integer satisfying

$$\frac{2a}{bn} + \frac{1}{n^2} < 2 - \frac{a^2}{b^2} \quad \leftarrow \text{HW show such an } n \text{ exists}$$

Then

$$\begin{aligned} \frac{a^2}{b^2} + \frac{2a}{bn} + \frac{1}{n^2} &< 2 \\ \Rightarrow \left(\frac{a}{b} + \frac{1}{n} \right)^2 &< 2 \end{aligned}$$

But $\frac{a}{b} + \frac{1}{n}$ is a rational number, so $\frac{a}{b} + \frac{1}{n} \in S$, yet $\frac{a}{b} < \frac{a}{b} + \frac{1}{n}$, and this contradicts that $\frac{a}{b}$ is an upper bound of S .

If $\frac{a}{b} \notin S$, so $(\frac{a}{b})^2 \geq 2$, consider any n such that

$$\frac{2a}{bn} - \frac{1}{n^2} \leq \frac{a^2}{b^2} - 2$$

and notice

$$\frac{2a}{bn} - \frac{1}{n^2} \leq \frac{a^2}{b^2} - 2$$

$$\Rightarrow -\frac{2a}{bn} + \frac{1}{n^2} \geq 2 - \frac{a^2}{b^2}$$

$$\Rightarrow \frac{a^2}{b^2} - \frac{2a}{bn} + \frac{1}{n^2} > 2$$

$$\Rightarrow \left(\frac{a}{b} - \frac{1}{n}\right)^2 \geq 2$$

Now let $s \in S$, so $s^2 < 2$. Notice

$$s^2 < \left(\frac{a}{b} - \frac{1}{n}\right)^2$$

Choosing n large enough, we may suppose $\frac{a}{b} - \frac{1}{n} > 1$. HW

We claim $s < \frac{a}{b} - \frac{1}{n}$. If not, $s \geq \frac{a}{b} - \frac{1}{n}$ and

we have

$$s^2 = s \cdot s \geq \left(\frac{a}{b} - \frac{1}{n}\right)^2$$

But this contradicts our assertion $s^2 < 2 < \left(\frac{a}{b} - \frac{1}{n}\right)^2$.

Hence we must have $\frac{a}{b} - \frac{1}{n}$ is a smaller upper bound, so $\frac{a}{b}$ is not the supremum of S .

Similarly, a set of rational numbers bounded below may not have a rational. Our definition of real numbers will overcome these deficiencies.

Here is our preliminary definition of the reals:

The set of real numbers is the ordered field satisfying the following completeness axiom: every set bounded above has a supremum.

We are defining the set of all real numbers as a set with a very particular collection of properties. Notice that it's not immediately clear that a set with these properties should exist. For that we'll need a construction of the real numbers which will come later.

Let's go ahead and notice, though, that we can go ahead and prove some interesting theorems by supposing the existence of real numbers.

Thm

If $x < \varepsilon$ for every $\varepsilon > 0$, then $x \leq 0$.

Pf

Let S be the set of all real numbers s such that $s < \varepsilon$ for all $\varepsilon > 0$. This set is bounded above: for example, 1 is an upper bound. To see this note $1 > 0$, so taking $\varepsilon = 1$, we see every element of S is no greater than 1. By the completeness axiom, S must have a supremum. We claim 0 is the supremum.

Suppose instead $\sup S > 0$. Notice $\frac{1}{2} \sup S > 0$ as well, but $\frac{1}{2} \sup S < \sup S$. This contradicts the definition of $\sup S$.

If $\sup S < 0$, then $\frac{1}{2} \sup S > \sup S$, but $\frac{1}{2} \sup S < 0$, so for every $\varepsilon > 0$,

$$\frac{1}{2} \sup S < 0 < \varepsilon$$

So $\frac{1}{2} \sup S \in S$, and again this contradicts the def'n of the supremum.

As $\sup S \neq 0$ and $\sup S \neq 0$, we have $\sup S = 0$.

Hence for every $s \in S$, $s \leq \sup S = 0$.

□

Cor

If $x \geq 0$ and $x < \varepsilon$ for all $\varepsilon > 0$, then $x = 0$. $\square$

Thm (Archimedean Property of the Reals) x and y

For every pair of positive reals, there exists a positive integer n such that $nx > y$.

Pf

Suppose this was not the case. Then for every integer n , $nx \leq y$. Thus the set

$$S = \{nx \mid n \in \mathbb{N}\}$$

is bounded above, and so has a supremum by the completeness axioms, so $nx \leq \sup S$ for every n . In particular, $2nx \leq \sup S$ for every n (since $2n$ is a positive integer). But then $nx \leq \frac{1}{2} \sup S$. But $\frac{1}{2} \sup S < \sup S$, and this contradicts the definition of $\sup$. $\square$

Cor

For every positive real x there exists an integer n such that $\frac{1}{n} < x$.

Pf

If not, $\frac{1}{n} \geq x$ for all n , and $1 \geq nx$

for every n . But this contradicts the Archimedean property above ($\gamma=1$ in the theorem above).

□

Lecture 05 - Odds and Ends

Before moving to the construction of the reals, we collect a few facts about rational and irrational numbers.

Lemma

If $m > n$ are integers and $m - n > 1$, then there exists an integer q with $n < q < m$.

Pf

Let $q = n + 1$. Then $n < q$ is obvious, and

$$m - q = m - (n + 1) = m - n - 1 > 1 - 1 > 0, \text{ so } m > q. \quad \square$$

Lemma

If $x < y$ are real numbers and $y - x > 1$, then there exists an integer n with $x < n < y$.

Pf

Note $\lceil y \rceil$ and $\lfloor x \rfloor$ are integers, and

$$\lceil y \rceil - \lfloor x \rfloor > y - x > 1$$

so there exists an integer n between $\lceil y \rceil$ and $\lfloor x \rfloor$. We claim such an n is b/w x and y .

WTS: $x < n < y$. If this were not the case, then n is an integer with $n \leq x$ or $n \geq y$.

But, as n is an integer, this would mean $n \leq \lfloor x \rfloor$ or $n \geq \lceil y \rceil$, contradicting that $\lfloor x \rfloor < n < \lceil y \rceil$. □

Lemma

Between any real numbers $x < y$, there exists a rational number.

Pf

Ob $x < y$, $y - x > 0$ and by the Archimedean property we can find a positive integer q so that $q(y - x) > 1$. By the lemma above, this means there's an integer p between qx and qy as $qy - qx = q(y - x) > 1$. That is, $qx < p < qy$, so $x < \frac{p}{q} < y$. □

Lemma

Between any pair of rational numbers there exists an irrational number.

Pf

First note $\frac{1}{\sqrt{2}}$ is irrational and $0 < \frac{1}{\sqrt{2}} < 1$. Let $\frac{p}{q} < \frac{r}{s}$ be rational numbers. Then $0 < \frac{r}{s} - \frac{p}{q}$. By the Archimedean property, there's an integer n so $n(\frac{r}{s} - \frac{p}{q}) > 1$. Thus

$$0 < \frac{1}{\sqrt{2}} < n \left(\frac{r}{s} - \frac{p}{q} \right)$$

$$\Rightarrow 0 < \frac{1}{n\sqrt{2}} < \frac{r}{s} - \frac{p}{q}$$

$$\Rightarrow \frac{p}{q} < \frac{p}{q} + \frac{1}{n\sqrt{2}} < \frac{r}{s}$$

The claim follows if $\frac{p}{q} + \frac{1}{n\sqrt{2}}$ is irrational, which is left as an exercise. $\square$

Cor of the two previous lemmas

Between any pair of real numbers there are infinitely-many rational and infinitely-many irrational numbers.

Pf

Exercise.

Even though there are infinitely-many rationals and irrationals between any two rationals, there are "more" irrationals.

Recall a set S is said to be countably infinite if there is a bijection $S \rightarrow \mathbb{Z}$. A set S is uncountable if there is an injective map $\mathbb{Z} \rightarrow S$

but ~~no~~ injective map $S \rightarrow \mathbb{Z}$. In this case S has "more" elements than $\mathbb{Z}$.

We claim the rationals are countable and the irrationals are uncountable. To establish the countability of the rationals, the following theorem will be helpful:

Thm

A countably infinite union of countably infinite sets is countable.

Pf

Let $S_1, S_2, S_3, \dots$ be a countably infinite collection of sets, each of which is countably infinite. So there are bijections $\varphi_i: \mathbb{Z} \rightarrow S_i$ for each i . Now we build an injection $\Phi: \bigcup_i S_i \rightarrow \mathbb{Z}$. Let p_i be the i^{th} prime ($p_1=2, p_2=3, p_3=5, p_4=7, p_5=11, \dots$)

Then define

$$\Phi(s) = p_i^{\varphi_i^{-1}(s)} \quad \text{if } s \in S_i$$

To see this is an injection, suppose

$$\Phi(s) = \Phi(s')$$

then

$$p_i^{\varphi_i^{-1}(s)} = p_{i'}^{\varphi_{i'}^{-1}(s')} \quad \text{if } s \in S_i, s' \in S_{i'}.$$

Since there are powers of primes, their equality means $p_i = p_{i'}$, so $i = i'$. Hence

$$\varphi_i^{-1}(s) = \varphi_i^{-1}(s')$$

But φ_i is a bijection, hence injective, so $s = s'$.

Any injection $\Psi: \mathbb{Z} \rightarrow \mathbb{Z}$ can be upgraded to a bijection $\Phi: \mathbb{Z} \rightarrow \mathbb{Z}$ as outlined in an upcoming homework assignment and that completes the proof. $\square$

Now we can use the previous theorem to easily prove $\mathbb{Q}$ is countably infinite

Thm

The set of rational numbers is countably infinite.

Pf
Let S_1 be the set of integers, thought of as fractions $n/\text{denominator } 1$.

$$S_1 = \left\{ \frac{n}{1} \mid n \in \mathbb{Z} \right\} = \{ \dots, -3, -2, -1, 0, 1, 2, 3, \dots \}$$

Let S_2 be the set of rational numbers which in lowest terms have denominator 2 — this just means 2 doesn't divide the numerator

$$S_2 = \left\{ \frac{n}{2} \mid n \in \mathbb{Z} \text{ and } 2 \nmid n \right\} = \left\{ \dots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \right\}$$

S_3 is the set of rational numbers with 3 as the denominator in lowest terms,

$$S_3 = \left\{ \frac{n}{3} \mid n \in \mathbb{Z} \text{ and } 3 \nmid n \right\} = \left\{ \dots, -\frac{4}{3}, -\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{7}{3}, \dots \right\}$$

In general, for each $d \in \mathbb{N}$, let S_d be the set of rational numbers which have denominator d in lowest terms

$$S_d = \left\{ \frac{n}{d} \mid n \in \mathbb{Z} \text{ and } \gcd(n, d) = 1 \right\}$$

Question for students
Why is this equiv. to $3 \nmid n$ and $2 \nmid n$ when $d=3$ or $d=2$?

Each S_d is obviously countable. Let $\varphi_d: S_d \rightarrow \mathbb{Z}$ be the injective map $\varphi_d\left(\frac{n}{d}\right) = n$.

As we have a countable collection of countable sets, their union is countable, but this union is exactly $\mathbb{Q}$ since every rational number can be written in lowest terms and so is an element of some S_d (i.e., $\mathbb{Q} \subseteq \bigcup_{d \in \mathbb{N}} S_d$), and every S_d consists of rational numbers by definition (each $S_d \subseteq \mathbb{Q} \Rightarrow \bigcup_d S_d \subseteq \mathbb{Q}$). □

The uncountability of the reals uses a famous trick called Cantor's diagonalization argument.

Then

There are uncountably many real numbers

Pf

Suppose instead there were countably many reals. Then there exists a bijection $\varphi: \mathbb{Z} \rightarrow \mathbb{R}$. Consider the decimal expansion of these real numbers. Say

$$\varphi(i) = \lfloor \varphi(i) \rfloor + 0.d_1^i d_2^i d_3^i \dots$$

Now we construct a number

$$\Delta = 0.\delta_1\delta_2\delta_3\cdots$$

by defining

$$\delta_i = d_i^i + 1 \pmod{10}$$

i.e., if $d_1^1 = 0, \delta_1 = 1$
 $d_2^2 = 1, \delta_2 = 2$
 $d_3^3 = 2, \delta_3 = 3$
 $\vdots$
 $d_i^i = 9, \delta_i = 0$

Notice $\Delta \neq \varphi(i)$ for any i by construction, hence φ can not be bijective. $\square$

Since the reals are a disjoint union of rationals and irrationals and the rationals are countable, the irrationals must be uncountable.

We'll end with a word on language and statements of a few properties about cardinality which we won't prove here for the sake of time, but which are shown in your textbook

We will adopt the convention that calling

a set countable means it is either finite or countably infinite.

Thm

- Every subset of a countable set is countable
- Every interval (a,b) (or $[a,b)$ or $(a,b]$ or $[a,b]$) contains uncountably irrational numbers and countably many rational numbers.

Lecture 06- Sequences, Pt. 1

Many objects and properties we'll care about later in the semester can be defined in terms of sequences. So, here we collect some basic definitions regarding sequences.

In general, given any set S , a sequence of elements of S is just a function $x: \mathbb{N} \rightarrow S$.

Often we think of a sequence as an infinite list and this is related to our definition by taking the first element of the list to be $x(1)$, the second element to be $x(2)$, and so on.

Often when x is a sequence we write $x_n = x(n)$.

Sometimes a sequence is defined by an expression

$$\left\{ \frac{2^n}{n^2} \right\}_{n=1}^{\infty} \quad \text{or} \quad \left(\frac{2^n}{n^2} \right)_{n=1}^{\infty}$$

This is really short-hand for

$$x: \mathbb{N} \rightarrow \mathbb{Q} \quad (\text{or } \mathbb{N} \rightarrow \mathbb{R})$$
$$x(n) = \frac{2^n}{n^2}$$

More generally we could consider functions whose domain is

$$\mathbb{Z}_{\geq m} = \{m+n \mid n \in \mathbb{N}_0\} = \{m, m+1, m+2, \dots\}$$

for some integer m . E.g., $\mathbb{Z}_{\geq 3} = \{3, 4, 5, \dots\}$
or $\mathbb{Z}_{\geq -7} = \{-7, -6, -5, \dots\}$.

It doesn't really matter if our sequence has domain $\mathbb{N}$ or $\mathbb{N}_0$ or $\mathbb{Z}_{\geq m}$ — they're all countably infinite sets with an "initial" element (1 for $\mathbb{N}$, 0 for $\mathbb{N}_0$, m for $\mathbb{Z}_{\geq m}$) and a notion of the "next" or "successor" element.

Every sequence $x: \mathbb{Z}_{\geq m} \rightarrow S$ can be re-interpreted as a sequence $x': \mathbb{N} \rightarrow S$ by defining

$$x'(n) = x(n-m-1)$$

Because it doesn't really matter whether our domain is $\mathbb{N}$ or $\mathbb{Z}_{\geq m}$, we'll usually just assume it's $\mathbb{N}$, but on odd occasions, $\mathbb{Z}_{\geq m}$ might be more convenient.

When defining a sequence we often like to have an explicit expression which tells us how to compute the n^{th} element. (e.g.,)

$$x_n = \frac{n^2 - 1}{2n^2 + n + 1} \quad \text{or} \quad \left\{ \frac{n^2 - 1}{2n^2 + n + 1} \right\}_{n=1}^{\infty} \quad \text{or} \quad x(n) = \frac{n^2 - 1}{2n^2 + n + 1}$$

Sometimes, though, we may define some elements in terms of other elements, such as

$$x_0 = 1$$

$$x_n = n \cdot x_{n-1} \quad \text{for } n \geq 1$$

In the last case we say our sequence is defined recursively or by a recurrence equation. Notice when giving such a sequence we need to explicitly give a few values (usually at the "beginning" of the sequence), otherwise our recursive computation would go on forever.

There are several bits of vocabulary we have for the different types of sequences we'll consider: We say a sequence of real numbers x_n is...

- bounded above if there exists an M such that $x_n \leq M$ for every n
- bounded below if there exists an m such that $x_n \geq m$ for every n
- bounded if it is both bounded above and bounded below. i.e., there exist m, M such that $m \leq x_n \leq M$ for every n .
- increasing if $x_n \leq x_{n+1}$ for every n .
- decreasing if $x_n \geq x_{n+1}$ for every n .

Notice that our definitions of increasing and decreasing allow for equality. As a consequence, any constant sequence is both increasing and decreasing. If we do not want to allow different terms of our sequence to be equal, we'll use the terms "strictly increasing" and "strictly decreasing"

- A sequence is strictly increasing if $x_n < x_{n+1}$ for every n
- A sequence is strictly decreasing if $x_n > x_{n+1}$ for every n .

Sometimes we will care that a sequence is increasing or decreasing, but not care which one it is, and here we use the term "monotonic"

- A sequence is monotonic if it is increasing or decreasing
- A sequence is strictly monotonic if it is strictly increasing or strictly decreasing.

Sequences naturally arise in calculus through approximations. For example, the area under a graph $y=f(x)$ for $a \leq x \leq b$ is approximated by considering a Riemann sum giving the areas of rectangles which fit into that region. We get better approximations by using more and more rectangles, and so we lead to the sequence of successive approximations.

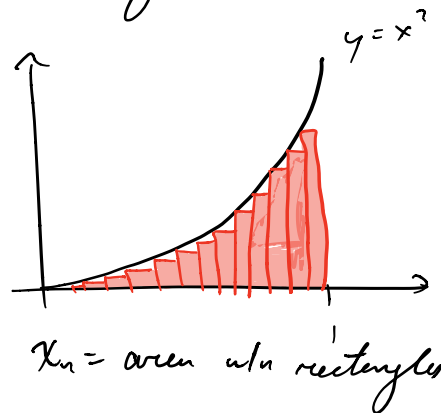
E.g., the area under $y=x^2$ over $[0,1]$ on the x -axis is approximated using n rectangles by

$$\sum_{k=1}^n \left(\frac{k}{n}\right)^2 \cdot \frac{1}{n} = \sum_{k=1}^n \frac{k^2}{n^3}$$

$$= \frac{2n^3 + 6n^2 + n}{6n^3}$$

We are thus lead to the sequence

$$x_n = \frac{2n^3 + 6n^2 + n}{6n^3}$$



Of course, what we'd like to do is see what happens as we use more and more and more rectangles — i.e., what happens as we go really, really, really far out into the sequence? Namely, do the elements of our sequence "approach" any particular value?

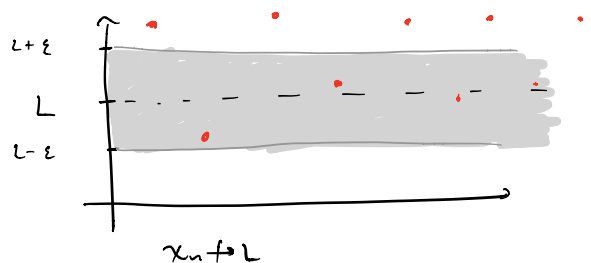
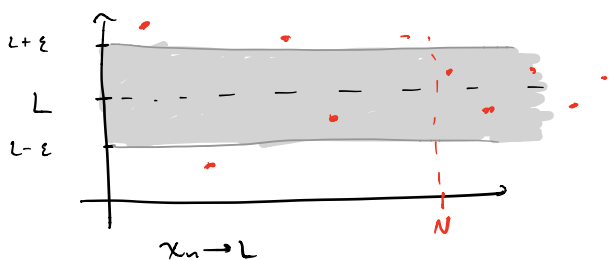
This leads us to consider the limit of a sequence.

We say a real number L is the limit of a sequence of real numbers x_n if for every $\varepsilon > 0$ there exists a number N such that $|x_n - L| < \varepsilon$ for all $n > N$. We then write $\lim_{n \rightarrow \infty} x_n = L$ and also say x_n converges

to L . This is also sometimes written $x_n \rightarrow L$.

The idea in this definition is that the elements of x_n get as close to L as you'd like — this is what the ϵ measures — and you stay that close to L , provided you go far enough out into the sequence. The N tells us how far we have to go out into the sequence.

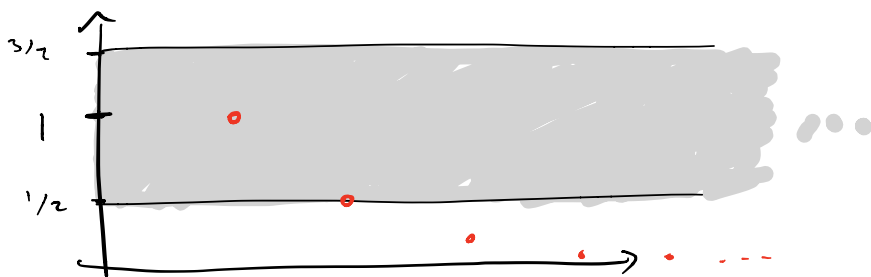
Before seeing an example of using this definition, it might be helpful to consider a "non-example." So, what would it mean if $x_n \not\rightarrow L$? This would mean there's some $\epsilon > 0$ such that for every choice of n there exists an $n > N$ with $|x_n - L| \geq \epsilon$. That is, no matter how far we go in the sequence, there are still going to be elements which aren't within ϵ -distance of L .



For example, consider the sequence $x_n = \frac{1}{n}$. Let's show this sequence does not converge to 1. If it did, then for every $\varepsilon > 0$ we could find an N such that $|1 - \frac{1}{n}| < \varepsilon$ for all $n > N$. However, if we take $\varepsilon = \frac{1}{2}$, notice that for all $n \geq 3$ we have

$$n \geq 3 \Rightarrow \frac{1}{3} \geq \frac{1}{n} \\ \Rightarrow -\frac{1}{3} \leq -\frac{1}{n} \Rightarrow 1 - \frac{1}{3} \leq 1 - \frac{1}{n}$$

$$\left|1 - \frac{1}{n}\right| = 1 - \frac{1}{n} > 1 - \frac{1}{3} = \frac{2}{3} > \frac{1}{2}$$



Thus $x_n \not\rightarrow 1$.

Note to self
Do $x_n \rightarrow 0$ example (below) first.

As a more interesting example, consider the sequence

$$x_n = \frac{n+2}{3n+1}$$

Let's show that this sequence converges to $\frac{1}{3}$.

To do this, we need to show that for every $\varepsilon > 0$ there exists an $N > 0$ such that if $n > N$, then $\left|\frac{n+2}{3n+1} - \frac{1}{3}\right| < \varepsilon$.

I claim taking $N = \frac{5-3\varepsilon}{9\varepsilon}$ will work (we'll see where this expression came from shortly):

Let $\varepsilon > 0$ be given, take $N = \frac{5-3\varepsilon}{9\varepsilon}$, and let $n > N$. Notice

$$\begin{aligned} n &> N \\ \Rightarrow n &> \frac{5-3\varepsilon}{9\varepsilon} \\ \Rightarrow 9n &> \frac{5-3\varepsilon}{\varepsilon} = \frac{5}{\varepsilon} - 3 \\ \Rightarrow 9n+3 &> \frac{5}{\varepsilon} \\ \Rightarrow \varepsilon &> \frac{5}{9n+3} = \frac{6-1}{9n+3} \\ &= \frac{3n+6-3n-1}{9n+3} \\ &= \frac{3n+6}{9n+3} - \frac{3n+1}{9n+3} \\ &= \frac{3}{3} \cdot \frac{n+2}{3n+1} - \frac{3n+1}{3n+1} \cdot \frac{1}{3} \\ &= \frac{n+2}{3n+1} - \frac{1}{3} \\ &= \left| \frac{n+2}{3n+1} - \frac{1}{3} \right| \end{aligned}$$

$3n+6 > 3n+1$
 $\Rightarrow 3(n+2) > 3n+1$
 $\Rightarrow \frac{n+2}{3n+1} > \frac{1}{3}$

Thus $\left| \frac{n+2}{3n+1} - \frac{1}{3} \right| < \varepsilon$ provided $n > N = \frac{5-3\varepsilon}{9\varepsilon}$,
and so $\frac{n+2}{3n+1} \rightarrow \frac{1}{3}$

Of course, the question you should ask yourself now is where $\frac{5-3\varepsilon}{9\varepsilon}$ came from. This came from "reverse-engineering" the problem: we start with what we want to end up with, $\left| \frac{n+2}{3n+1} - \frac{1}{3} \right| < \varepsilon$, and work our way backwards to see what this would require of n :

$$\left| \frac{n+2}{3n+1} - \frac{1}{3} \right| < \varepsilon$$

$$\Rightarrow \frac{n+2}{3n+1} - \frac{1}{3} < \varepsilon$$

$$\Rightarrow \frac{3}{3} \cdot \frac{n+2}{3n+1} - \frac{3n+1}{3n+1} \cdot \frac{1}{3} < \varepsilon$$

$$\Rightarrow \frac{3n+6-3n-1}{9n+3} < \varepsilon$$

$$\Rightarrow \frac{6-1}{9n+3} < \varepsilon$$

$$\Rightarrow \frac{5}{9n+3} < \varepsilon$$

$$\Rightarrow \frac{5}{\varepsilon} < 9n+3$$

$$\Rightarrow \frac{5}{\varepsilon} - 3 < 9n$$

$$\Rightarrow \frac{5-3\varepsilon}{\varepsilon} < 9n$$

$$\Rightarrow \frac{5-3\varepsilon}{9\varepsilon} < n.$$

Notice the steps in our pf $\frac{n+2}{3n+1} \rightarrow \frac{1}{3}$ above basically consisted of these steps in reverse.

As another example, let's show $\frac{1}{n} \rightarrow 0$. We want to show for every $\varepsilon > 0$, there exists an N such that if $n > N$, then $|\frac{1}{n} - 0| < \varepsilon$.

First note our goal, $|\frac{1}{n} - 0| < \varepsilon$, is equivalent to $\frac{1}{n} < \varepsilon$ which is equivalent to $1 < \varepsilon n$. The Archimedean property promises us there is an integer N so that $N\varepsilon > 1$. If we take $n > N$, then

$$n\varepsilon > N\varepsilon > 1 \Rightarrow \frac{1}{n} < \varepsilon$$

So $\frac{1}{n} \rightarrow 0$.

A very similar computation can show, for example, $\frac{1}{\sqrt{n}} \rightarrow 0$ as well, and will be left as an exercise.

For a more interesting example, let's show $\frac{n^2+1}{n^2+2} \rightarrow 1$. We want to show for every $\varepsilon > 0$, there exists an $N > 0$ such that $|\frac{n^2+1}{n^2+2} - 1| < \varepsilon$ for every $n > N$. Let's again "reverse engineer" the problem:

$$\left| \frac{n^2+1}{n^2+2} - 1 \right| = \left| -\frac{n^2+1}{n^2+2} \right| < \varepsilon$$

$$\Rightarrow \frac{n^2+2 - n^2 - 1}{n^2+2} < \varepsilon$$

$$\Rightarrow \frac{1}{n^2+2} < \varepsilon$$

$$\Rightarrow \frac{1}{\varepsilon} < n^2+2$$

$$\Rightarrow \frac{1}{\varepsilon} - 2 < n^2$$

Now, it's tempting to write $n > \sqrt{\frac{1}{\varepsilon} - 2}$, but we have a problem: what if $\frac{1}{\varepsilon} - 2$ is negative? We can fix this by defining N piecewise:

$$N = \begin{cases} \sqrt{\frac{1}{\varepsilon} - 2} & \text{if } \frac{1}{\varepsilon} > 2 \\ 1 & \text{otherwise} \end{cases}$$

Notice in the latter case we'd have, for $n > N$,

$$n^2 > n > 1 > 0 > \frac{1}{\varepsilon} - 2$$

Now we can complete our pf of $\frac{n^2+1}{n^2+2} \rightarrow 1$:

Let $\varepsilon > 0$ be given and take

$$N = \begin{cases} \sqrt{\frac{1}{\varepsilon} - 2} & \text{if } \frac{1}{\varepsilon} > 2 \\ 1 & \text{otherwise} \end{cases}$$

Let $n > N$. If $\frac{1}{\varepsilon} \leq 2$, then $n > 1$ so
 $n^2 > n > 1 > 0 > \frac{1}{\varepsilon} - 2$. If $\frac{1}{\varepsilon} > 2$, then $\sqrt{\frac{1}{\varepsilon} - 2}$
 is defined and $n > N = \sqrt{\frac{1}{\varepsilon} - 2}$ implies $n^2 > \frac{1}{\varepsilon} - 2$.
 That is, for all ε our choice of N guarantees
 $n^2 > \frac{1}{\varepsilon} - 2$ for $n > N$. Thus

$$\begin{aligned}
 & n^2 > \frac{1}{\varepsilon} - 2 \\
 \Rightarrow & n^2 + 2 > \frac{1}{\varepsilon} \\
 \Rightarrow & \frac{1}{n^2 + 2} < \varepsilon \\
 \Rightarrow & \frac{2 - 1}{n^2 + 2} < \varepsilon \\
 \Rightarrow & \frac{n^2 + 2 - n^2 - 1}{n^2 + 2} < \varepsilon \\
 \Rightarrow & 1 - \frac{n^2 + 1}{n^2 + 2} < \varepsilon \\
 \Rightarrow & \left| \frac{n^2 + 1}{n^2 + 2} - 1 \right| < \varepsilon
 \end{aligned}$$

Where the last step follows from

$$\begin{aligned}
 & n^2 + 1 < n^2 + 2 \\
 \Rightarrow & \frac{n^2 + 1}{n^2 + 2} < 1 \\
 \Rightarrow & \frac{n^2 + 1}{n^2 + 2} - 1 < 0 \\
 \Rightarrow & \left| \frac{n^2 + 1}{n^2 + 2} - 1 \right| = - \left(\frac{n^2 + 1}{n^2 + 2} - 1 \right) = 1 - \frac{n^2 + 1}{n^2 + 2}
 \end{aligned}$$

Thus $\frac{n^2+1}{n^2+2} \rightarrow 1$.

In principle we can always use this formal ε -definition to justify why the limit of a sequence is the value we expect it to be, but in practice it's rather tedious. Thus it would be convenient to build up some theory that removed the burden of using this definition directly - at least some of the time.

First let's establish some basic properties of sequences and their limits.

Thm

Every increasing sequence bounded above converges.
Pf

Suppose x_n is an increasing sequence bounded above. Notice the set $S = \{x_n \mid n \in \mathbb{N}\}$ is bounded above, so by the completeness axiom has a supremum - call it σ . We claim $x_n \rightarrow \sigma$.

For every n , note $|x_n - \sigma| = \sigma - x_n$ since σ is an upperbound for the set S above.

Let $\varepsilon > 0$ be given. From an exercise (Problem 6 on Homework 2) we know that since σ is a supremum, there exists an $s \in \{x_n \mid n \in \mathbb{N}\}$ such that $s > \sigma - \varepsilon$. Note

$$\begin{aligned} s &> \sigma - \varepsilon \\ \Rightarrow -s &< \varepsilon - \sigma \\ \Rightarrow \sigma - s &< \sigma + \varepsilon - \sigma = \varepsilon \end{aligned}$$

But $s = x_N$ for some N . If $n > N$, then $x_n > x_N$ because the sequence is increasing, so for all $n > N$ we have

$$\begin{aligned} x_n &> x_N \\ \Rightarrow \sigma - x_n &< \sigma - x_N < \varepsilon \\ \Rightarrow |\sigma - x_n| &< \varepsilon \end{aligned}$$

↓ This is $\sigma - s < \varepsilon$ above

Hence $x_n \rightarrow \sigma = \sup \{x_n \mid n \in \mathbb{N}\}$. □

Thm

Every decreasing sequence bounded below converges.

Pf

Homework.

As a sort of "partial converse," we have the following statement

Thm

Every convergent sequence is bounded.

The key ingredient in proving this theorem is the observation that a convergent sequence must be "eventually bounded" — almost by definition since the sequence eventually must live in the interval $(L-\epsilon, L+\epsilon)$ for every $\epsilon > 0$ if the limit is L . There's some cut-off N such that for x_n with $n > N$ the sequence $x_{n > N}$ is bounded above by $L+\epsilon$ and below by $L-\epsilon$. But notice there are only finitely-many elements that come before N , and any finite set is necessarily bounded (bounded above by the maximum element, and bounded below by the minimum element).

(A more detailed proof starts on pg. 54 of Gordon.)

With our definition of the limit of a sequence established, we can now prove a useful theorem from calculus:

Thm

Suppose x_n and y_n are convergent sequences of real numbers with $x_n \rightarrow L$ and $y_n \rightarrow M$.

- $kx_n \rightarrow kL$ for any constant k
- $x_n + y_n \rightarrow L + M$
- $x_n - y_n \rightarrow L - M$
- $x_n y_n \rightarrow LM$
- $\frac{x_n}{y_n} \rightarrow \frac{L}{M}$, provided $M \neq 0$ and no y_n equals zero.

Pf

We'll only prove $x_n + y_n \rightarrow L + M$. The book proves $x_n y_n \rightarrow LM$; the other properties will follow from these two.

Let $\varepsilon > 0$ be given. Since $x_n \rightarrow L$, $\exists N$ st.
 $|x_n - L| < \varepsilon/2$ for $n > N$, and similarly $\exists P$ st.
 $|y_n - M| < \varepsilon/2$ for $n > P$. Notice for all $n > \max\{N, P\}$,

$$\begin{aligned} |x_n + y_n - (L + M)| &= |x_n - L + y_n - M| \leq |x_n - L| + |y_n - M| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

□

We say the limit of a sequence x_n does not exist (DNE) if for every real number L , $x_n \not\rightarrow L$. For example, $x_n = (-1)^n$ does not converge to any real number.

As a special case, we say x_n converges to infinity, denoted $x_n \rightarrow \infty$, if for every $M \in \mathbb{R}$, $\exists N$ st. $x_n > M$ for all $n > N$.

As an easy example, consider $x_n = n^2$. Given any M , we define

$$N = \begin{cases} 2 & \text{if } M \leq 1 \\ \sqrt{M} & \text{if } M > 1 \end{cases}$$

If $M \leq 1$, then for any $n > N$, we have $n > 2$, so $x_n = n^2 > 2^2 = 4 > M$. If $M > 1$, then for $n > N$ we have $n > \sqrt{M}$, so

$$x_n = n^2 > (\sqrt{M})^2 = M.$$

Hence $n^2 \rightarrow \infty$.

Lecture 07- Sequences, Pt. 2

Sometimes the algebraic manipulations required to show a particular sequence converges using the ϵ -definition are difficult or non-obvious. In these cases it's often convenient to replace our sequence with simpler ones. The following theorem is often useful in these situations.

Thm (Squeeze Theorem / Sandwich Theorem)

If x_n, a_n, b_n are sequences such that $a_n \leq x_n \leq b_n$ for all "sufficiently large n " (i.e., these inequalities may not hold for every n , but they do hold for all $n > M$ for some M) and if $a_n \rightarrow L$ and $b_n \rightarrow L$, then $x_n \rightarrow L$ too.

Pf
Let $\epsilon > 0$ be given. As $a_n \rightarrow L$, there exists an N_a such that $|a_n - L| < \epsilon$ for all $n > N_a$. Similarly, there exists an N_b such that $|b_n - L| < \epsilon$ for all $n > N_b$. Note we may write this as

$$-\epsilon < a_n - L < \epsilon \Rightarrow L - \epsilon < a_n < L + \epsilon$$

$$-\epsilon < b_n - L < \epsilon \Rightarrow L - \epsilon < b_n < L + \epsilon$$

Let $N = \max\{N_a, N_b, M\}$ (M as in the comment in the statement of the theorem). Then

for all $n > N$ we have

$$L - \varepsilon < a_n \leq x_n \leq b_n < L + \varepsilon \\ \Rightarrow |x_n - L| < \varepsilon.$$

□

For example, consider

$$x_n = \frac{2n + (-1)^n}{6n + 5}$$

Notice $-1 \leq (-1)^n \leq 1$, hence

$$2n - 1 \leq 2n + (-1)^n \leq 2n + 1$$

So

$$\frac{2n-1}{6n+5} \leq \frac{2n + (-1)^n}{6n+5} \leq \frac{2n+1}{6n+5}$$

We can show $\frac{2n-1}{6n+5}$ and $\frac{2n+1}{6n+5}$ both converge to $\frac{1}{3}$, so by the squeeze theorem, $\frac{2n + (-1)^n}{6n+5} \rightarrow \frac{1}{3}$ as well.

It's tempting to suppose if x_n is a convergent sequence and $x_n < \beta$ for all n , then the

limit of x_n - call it L - also satisfies $L < \beta$, but this isn't quite true. For example, consider

$$x_n = 1 - \frac{1}{10^n} = \underbrace{0.999\dots 9}_{n \text{ 9's}}$$

Every x_n satisfies $x_n < 1$, yet the limit actually equals 1. In general we have the following:

Prop

If $x_n \rightarrow L$ and $x_n < \beta$ for all n , then $L \leq \beta$.

Pf

We show L can not be larger than β .

If $L > \beta$, let $\varepsilon = L - \beta$. By assumption, $x_n \rightarrow L$ so $\exists N$ st. $|x_n - L| < \varepsilon$ for all $n > N$.

As $L > \beta > x_n$, $|x_n - L| = L - x_n$, so

$$\begin{aligned} |x_n - L| &< \varepsilon \\ \Rightarrow L - x_n &< \varepsilon \\ \Rightarrow L - x_n &< L - \beta \end{aligned}$$

$$\begin{aligned} \Rightarrow -x_n < -\beta \\ \Rightarrow x_n > \beta \end{aligned}$$

But this contradicts our assumption $x_n < \beta$. $\square$

The following Proposition is proved in almost the same way.

Prop.

If $x_n \rightarrow L$ and $x_n > \alpha$ for all n , then $L \geq \alpha$.

Pf

Exercise. $\square$

As a corollary of the two propositions we have

Cor

If x_n is a convergent sequence with $x_n \in (\alpha, \beta)$ for all n , then the limit L must occur in $[\alpha, \beta]$. The same conclusion holds if $x_n \in [\alpha, \beta]$, $x_n \in [\alpha, \beta)$, or $x_n \in (\alpha, \beta]$. $\square$

The main take-away from this corollary is that

when taking limits, strict inequalities ($<$ and $>$) may fail to be strict (they get replaced by $\leq$ or $\geq$).

Using the ϵ -definition of limit requires we know what the limit of the sequence is supposed to be. Often this is difficult and so we want some other criteria that guarantee convergence. As we've seen, an increasing (respectively, decreasing) sequence which is bounded above (resp, bounded below) must converge. But lots of sequences we care about aren't monotonic, so we need some more general criteria. A stepping stone in this direction is a collection of theorems concerning subsequences — under some very general condition we can guarantee a sequence has a convergent subsequence, and that suffices for many arguments in analysis and related branches of mathematics such as dynamical systems and topology. Before giving these general criteria, though, let's be clear about what a "subsequence" is.

Let x_n be any sequence of real numbers

and let n_i be a strictly increasing sequence of natural numbers. Then x_{n_i} is a subsequence of x_n . To be more precise, let's interpret these as functions. Our first sequence, x_n , is really a function

$$x: \mathbb{N} \rightarrow \mathbb{R}$$

then we have a ^{strictly increasing} sequence of natural numbers,

$$n: \mathbb{N} \rightarrow \mathbb{N} \quad n(i+1) > n(i).$$

The subsequence x_{n_i} is just the composition

$$x \circ n: \mathbb{N} \rightarrow \mathbb{R}$$

$$x_{n_i} = x(n(i)).$$

For example, if $x_n = \frac{n+2}{3n+1}$ and $n_i = i^2$, our subsequence is

$$x_{n_i} = \frac{i^2 + 2}{3i^2 + 1}$$

As lists of numbers,

$$x_n = \frac{3}{4}, \frac{4}{7}, \frac{5}{10}, \frac{6}{13}, \frac{7}{16}, \frac{8}{19}, \frac{9}{22}, \frac{10}{25}, \frac{11}{28}, \frac{12}{31}, \dots$$

$$n_i = 1, 4, 9, 16, 25, \dots$$

$$x_{n_i} = \frac{3}{4}, \frac{6}{13}, \frac{11}{28}, \dots$$

That is, x_{n_i} contains the 1st element of x_n , the 4th element of x_n , the 9th element of x_n , and so on.

Let's notice that even if a sequence doesn't have a limit, it may have a subsequence which does converge. For instance, consider

$$x_n = (-1)^n$$

$$n_i = 2i$$

then

$$x_{n_i} = (-1)^{2i} = 1$$

So, $x_n = -1, 1, -1, 1, -1, 1, -1, \dots$ which does not converge, but our subsequence $x_{n_i} = 1, 1, 1, 1, \dots$

does converge. If our original sequence x_n does converge, however, then all subsequences must converge to the same limit.

Thm

If $x_n \rightarrow L$, then $x_{n_i} \rightarrow L$ as well for every subsequence x_{n_i} .

Pf

Let $\varepsilon > 0$ be given. As $x_n \rightarrow L$, $\exists N$ st. $|x_n - L| < \varepsilon$ for all $n > N$. As n_i is a strictly increasing seq. of pos. integers, $\exists I$ st. $n_i > N$ for all $i > I$. Hence $|x_{n_i} - L| < \varepsilon$ for all $i > I$. □

Cor

If x_n has subsequences which converge to different values, $x_{n_i} \rightarrow L$ and $x_{n_j} \rightarrow M$ with $L \neq M$, then x_n does not converge.

We ultimately want to show that every bounded sequence has a convergent subsequence. Since every bounded monotonic sequence converges, it would suffice to show every (bounded)

sequence has a monotonic subsequence, but we can actually prove something stronger:

Thm

Every sequence has a monotonic subsequence

Pf
The idea of the proof will be to construct a decreasing subsequence, and if we can't we'll see there must be an increasing subsequence.

To begin, let's define n_1 to be the smallest value of $n_1 \in \mathbb{N}$ such that there exists inf. many $m > n_1$ with $x_{n_1} > x_m$. Then define n_2 to be the smallest $n_2 > n_1$ such that there exists inf. many $m > n_2$ such that $x_{n_1} > x_{n_2} > x_m$. In general, we continue to define n_k as the least integer n such that $n_k > n_{k-1}$ and such that there exist inf. many $m > n_k$ with $x_{n_{k-1}} > x_{n_k} > x_m$.

If we can do this forever, generating an infinite list $n_1 < n_2 < n_3 < \dots$, then by definition

we have a decreasing subsequence

$$x_{n_1} > x_{n_2} > x_{n_3} > \dots$$

If we can't do this forever, then our list is finite. Pick any m_1 larger than the last n_k (or $m_1 = 1$ if our list was empty).

Notice there are only finitely many x_p with $p > m_1$ but $x_p < x_{m_1}$ (otherwise our n_k list would have included m_1). Let m_2 be the least integer $m_2 > m_1$ where there exist inf. many x_p st. $x_p \geq x_{m_2} \geq x_{m_1}$. In general, we define x_{m_k} to be the least integer $m_k > m_{k-1}$ st. $\exists$ inf. many $p > m_k$ with $x_p \geq x_{m_k} \geq x_{m_{k-1}}$. Then $x_{m_1} \leq x_{m_2} \leq x_{m_3} \leq \dots$ is our incr. subseq.

Cor (Bolzano - Weierstrass Theorem)

Every bounded seq. has a convergent subsequence.
Pf

Our bounded seq. has a monotonic subsequence, which is inherently also bounded, so converges. $\square$

So, when a sequence is bounded it must have a convergent subsequence; if the sequence converges then all subsequences share a common limit, but what happens for bounded sequences that don't converge—what can we say about the possible limits of subsequences?

To study this we'll introduce two new concepts. Given any sequence x_n , we can associate two particular sequences that we can think of as the subsequences of largest and smallest elements. (Though these may not actually be subsequences of x_n .)

We'll define a sequence L_n as

$$L_n = \sup \{ x_n, x_{n+1}, x_{n+2}, \dots \}$$

and a sequence ω_n as

$$\omega_n = \inf \{ \omega_n, \omega_{n+1}, \omega_{n+2}, \dots \}$$

Notice the L_n sequence only makes sense if

x_n is bounded above, and ω_n only makes sense if x_n is bounded below.

E.g., Consider the sequence

$$x_n = 2 + \frac{(-1)^n}{n}$$

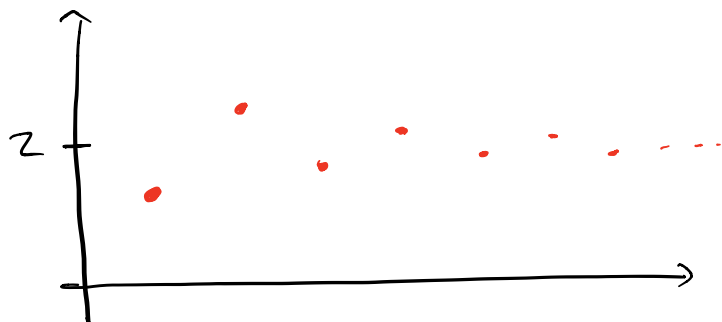
which has entries

$$2 - \frac{1}{2}, 2 + \frac{1}{3}, 2 - \frac{1}{4}, 2 + \frac{1}{5}, 2 - \frac{1}{6}, 2 + \frac{1}{7}, \dots$$

or simply

$$\frac{3}{2}, \frac{7}{3}, \frac{7}{4}, \frac{11}{5}, \frac{11}{6}, \frac{15}{7}, \dots$$

Which we'll visualize as

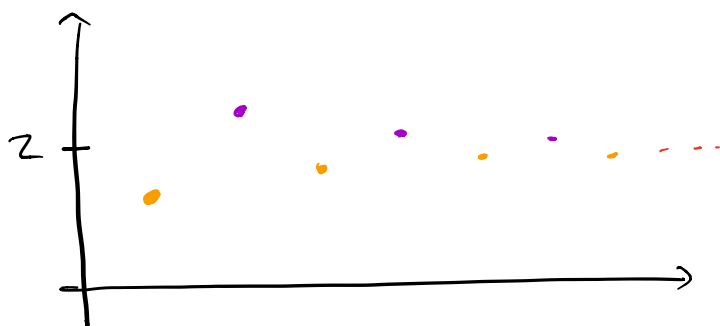


Let's note the x_n sequence is

$$2 + \frac{1}{3}, 2 + \frac{1}{5}, 2 + \frac{1}{7}, 2 + \frac{1}{9}, \dots$$

and the ω_n sequence is

$$2 - \frac{1}{2}, 2 - \frac{1}{4}, 2 - \frac{1}{6}, 2 - \frac{1}{8}, \dots$$



Let's notice these sequences are both monotonic

Lemma

Let x_n be a bounded sequence and α_n, ω_n the sequences $\alpha_n = \sup\{x_{n+k} \mid k \in \mathbb{Z}_{\geq 0}\}$, $\omega_n = \inf\{x_{n+k} \mid k \in \mathbb{Z}_{\geq 0}\}$. Then α_n is decreasing and ω_n is increasing.

Pf

We'll prove α_n is decreasing and leave the proof that ω_n is increasing as an exercise.

Notice for all $m > n$,

$$\underbrace{\{x_n, x_{n+1}, x_{n+2}, \dots\}}_{X_n} \supseteq \underbrace{\{x_m, x_{m+1}, x_{m+2}, \dots\}}_{X_m}$$

Thus every upper bound of X_n is an upper bound of X_m , so $\sup X_n$ is an upper bound for X_m and so $\sup X_m$ is no larger than $\sup X_n$, hence $\alpha_n \geq \alpha_m$. □

If x_n is bounded, then the above lemma tells us that α_n and ω_n must converge. Their limits have special names: the limit of α_n is called the limit superior (or just "lim-sup") of x_n and denoted $\limsup_{n \rightarrow \infty} x_n$; the limit of ω_n is the limit inferior (or "lim-inf") of x_n and denoted $\liminf_{n \rightarrow \infty} x_n$.

For our $x_n = 2 + \frac{(-1)^n}{n}$ sequence above we have

$$\limsup_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n = 2$$

In general, we have the following:

Then

A sequence x_n converges to L iff the liminf and limsup of x_n converge to L .

Pf

Suppose $x_n \rightarrow L$. Then $\forall \varepsilon > 0, \exists N$ st. $|x_n - L| < \varepsilon$ for all $n > N$. That is, $L - \varepsilon < x_n < L + \varepsilon$ for all $n > N$. Hence $\alpha_N = \sup \{x_N, x_{N+1}, \dots\} \leq L + \varepsilon$ and $\omega_N = \inf \{x_N, x_{N+1}, \dots\} \geq L - \varepsilon$ for all $n > N$.

We may write this as

$$\alpha_N \leq L + \varepsilon \Rightarrow \alpha_N - L \leq \varepsilon$$

$$\omega_N \geq L - \varepsilon \Rightarrow \omega_N - L \geq -\varepsilon \Rightarrow L - \omega_N \leq \varepsilon$$

As this holds for all $\varepsilon > 0$ (for sufficiently large N), we have $\alpha_N - L$ and $L - \omega_N$ converge to 0, so α_N and ω_N converge to L .

The converse will be left as an exercise. $\square$

Lecture 08 - Cauchy Sequences

We finish up our discussion of sequences by discussing a special type of sequence which is closely related to the definition of convergence of a sequence and outlining how these sequences can be used to define the real numbers constructively.

First, though, recall that our definition of convergence of a sequence can be difficult to use because it requires us to know the limit of the sequence. In some special cases we have some theorems that allow us to sidestep this issue, but these are usually very special cases. E.g., convergence of a bounded, monotonic sequence.

What we will describe now is something that is equivalent to convergence, but doesn't require us to know the limit.

We say that a sequence x_n is a Cauchy sequence if for every $\epsilon > 0$ there exists an N such that for every $m, n > N$ we have $|x_m - x_n| < \epsilon$.

That is, if a sequence is Cauchy, all its terms eventually get close to one another.

E.g., the sequence

$$x_n = \frac{n+1}{2n-3}$$

is Cauchy. To see this, we need to produce an $N > 0$ so that for a given $\epsilon > 0$, $\left| \frac{m+1}{2m-3} - \frac{n+1}{2n-3} \right| < \epsilon$ for all $m, n > N$. We claim taking $N = \frac{5+3\epsilon}{2\epsilon}$ works:
Let

$$m, n > N = \frac{5+3\epsilon}{2\epsilon}$$

Now notice

$$\left| \frac{m+1}{2m-3} - \frac{n+1}{2n-3} \right|$$

$$= \left| \frac{m+1}{2m-3} - \frac{1}{2} + \frac{1}{2} - \frac{n+1}{2n-3} \right|$$

$$\leq \left| \frac{m+1}{2m-3} - \frac{1}{2} \right| + \left| \frac{1}{2} - \frac{n+1}{2n-3} \right|$$

$$= \left| \frac{2(m+1) - (2m-3)}{2 \cdot (2m-3)} \right| + \left| \frac{2n-3 - 2(n+1)}{2 \cdot (2n-3)} \right|$$

$$= \left| \frac{2m+2-2m+3}{4m-6} \right| + \left| \frac{2n-3-2n-2}{4n-6} \right|$$

$$= \frac{5}{|4m-6|} + \frac{5}{|4n-6|}$$

$$< \frac{5}{\left| 4 \cdot \frac{5+3\varepsilon}{2\varepsilon} - 6 \right|} + \frac{5}{\left| 4 \cdot \frac{5+3\varepsilon}{2\varepsilon} - 6 \right|}$$

$$= \frac{10}{\left| \frac{4(5+3\varepsilon) - 6 \cdot 2\varepsilon}{2\varepsilon} \right|}$$

$$= \frac{20\varepsilon}{|20 + 12\varepsilon - 12\varepsilon|}$$

$$= \varepsilon.$$

The important thing about a Cauchy sequence is the following:

Thm

A sequence of real numbers x_n converges iff it is Cauchy.

Pf

Suppose $x_n \rightarrow L$. Then for every $\varepsilon > 0$ there exists an N such that $|x_n - L| < \frac{\varepsilon}{2}$ for all $n > N$.
Let m and n both be greater than this N .

Then

$$\begin{aligned} |x_m - x_n| &= |x_m - L + L - x_n| \\ &\leq |x_m - L| + |x_n - L| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Thus our convergent sequence x_n is Cauchy.

Now suppose x_n is a Cauchy sequence.

Notice there exists an N such that $|x_m - x_n| < 1$ for all $m, n > N$. Thus

$$\begin{aligned} |x_{N+1} - x_n| &< 1 \\ \Rightarrow -1 &< x_{N+1} - x_n < 1 \\ \Rightarrow -x_{N+1} - 1 &< -x_n < -x_{N+1} + 1 \\ \Rightarrow x_{N+1} - 1 &< x_n < x_{N+1} + 1 \end{aligned}$$

for all $n > N$. This implies the sequence is

bounded above by $\max\{x_1, x_2, \dots, x_N, x_{N+1} + 1\}$ and bounded below by $\min\{x_1, x_2, \dots, x_N, x_{N+1} - 1\}$. Hence the liminf and limsup of the sequence exist. We claim they are equal. If not, say

$$\limsup_{n \rightarrow \infty} x_n - \liminf_{n \rightarrow \infty} x_n =: \varepsilon$$

But there's an N st. $|x_m - x_n| < \varepsilon/2$ for all $m, n > N$.

Note, though

$$\limsup_{n \rightarrow \infty} x_n - \liminf_{n \rightarrow \infty} x_n$$

$$\leq \sup\{x_n | n > N\} - \inf\{x_n | n > N\}$$

$$\leq \varepsilon/2$$

$$\begin{aligned} \limsup_{n \rightarrow \infty} x_n &\leq \sup\{x_n | n \in \mathbb{N}\} \\ \liminf_{n \rightarrow \infty} x_n &\geq \inf\{x_n | n \in \mathbb{N}\} \end{aligned}$$

and this is a contradiction, so $\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n$, and the limit of the sequence exists. □

Question for the class

If a sequence has the property that for every $\varepsilon > 0$ $\exists N$ st. $|x_{n+1} - x_n| < \varepsilon$ for all $n > N$, does that mean x_n is Cauchy? i.e., if the difference of successive terms goes to zero, does that mean the sequence is Cauchy?

Your first thought might be "yes," but you have to compare a term to all remaining terms to be Cauchy. So, for example x_{n+2} and x_n may be as large as 2ε as:

$$\begin{aligned} |x_{n+2} - x_n| &= |x_{n+2} - x_{n+1} + x_{n+1} - x_n| \\ &\leq |x_{n+2} - x_{n+1}| + |x_{n+1} - x_n| \\ &< \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

Similarly, $|x_{n+3} - x_n| < 3\varepsilon$; $|x_{n+k} - x_n| < k\varepsilon$. So, as we look further out into the sequence, the distances could be getting large.

To be a bit more concrete, consider the sequence

$$x_n = \sum_{k=1}^n \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

Notice

$$|x_{n+1} - x_n| = \left| \sum_{k=1}^{n+1} \frac{1}{k} - \sum_{k=1}^n \frac{1}{k} \right| = \frac{1}{n+1}$$

Thus $|x_{n+1} - x_n| \rightarrow 0$. However, the x_n 's are the partial sums of the harmonic series, so they

diverge, and so x_n is not Cauchy.

Given the previous result, the following may be surprising:

Lemma

Let x_n be a sequence of real numbers such that there exists an M so that for all $n > M$, $|x_{n+1} - x_n| < \frac{1}{2^n}$. Then x_n is Cauchy.

Pf

We want to show that for every $\varepsilon > 0$ there is an N such that $|x_m - x_n| < \varepsilon$ for all $m, n > N$. So, let $\varepsilon > 0$ be given.

By the Archimedean property, there is an integer N such that $1 < 2^{N-1} \cdot \varepsilon$.

[Really the Archimedean property tells us $\exists k$ st. $1 < k\varepsilon$, but we'll just replace that k by a larger power of 2 and write that power of 2 as 2^{N-1} .] Making N larger if necessary, we may suppose $N > M$. Notice $\frac{1}{2^{N-1}} < \varepsilon$.

Now consider two $m, n > N$. Suppose without

loss of generality that $m > n$. [If $m = n$, then $x_m = x_n$ so $|x_m - x_n| = 0 < \epsilon$. So, one of m or n is larger, and we'll just suppose it's m .]

Let $l = m - n$, so $m = n + l$. Then

$$|x_m - x_n| = |x_{n+l} - x_n|$$

$$= |x_{n+l} - x_{n+l-1} + x_{n+l-1} - x_n|$$

$$= |x_{n+l} - x_{n+l-1} + x_{n+l-1} - x_{n+l-2} + x_{n+l-2} - x_n|$$

$$= \dots$$

$$= |x_{n+l} - x_{n+l-1} + x_{n+l-1} - x_{n+l-2} + x_{n+l-2} - x_{n+l-3} + \dots + x_{n+1} - x_n|$$

$$\leq |x_{n+l} - x_{n+l-1}| + |x_{n+l-1} - x_{n+l-2}| + \dots + |x_{n+1} - x_n|$$

$$< \frac{1}{2^{n+l-1}} + \frac{1}{2^{n+l-2}} + \frac{1}{2^{n+l-3}} + \dots + \frac{1}{2^n}$$

$$= \frac{1}{2^n} \left(\frac{1}{2^{l-1}} + \frac{1}{2^{l-2}} + \dots + \frac{1}{2^0} \right)$$

$$= \frac{1}{2^n} \cdot \sum_{i=0}^{l-1} \left(\frac{1}{2} \right)^i = \frac{1}{2^n} \cdot \frac{1 - \left(\frac{1}{2} \right)^l}{1 - \left(\frac{1}{2} \right)}$$

Recall the geometric sum formula,
 $\sum_{i=0}^{l-1} r^i = \frac{1-r^l}{1-r}$

$$= \frac{1}{2^n} \cdot 2 \cdot \left(\frac{2^L - 1}{2^L} \right) < \frac{1}{2^{n-1}}$$

$$< \frac{1}{2^{N-1}} < \varepsilon.$$

□

So, having successive terms go to zero in general doesn't imply a sequence is Cauchy, unless the differences shrink to zero "quickly."

Question for students

The $\frac{1}{2^n}$ in the lemma above isn't magical. What other values would give the same result?

Example

Show the sequence $x_n = \sum_{k=0}^{n-1} \frac{2^k - k}{k3^{k+1}}$ is Cauchy.

Note

$$|x_{n+1} - x_n| = \left| \sum_{k=0}^n \frac{2^k - k}{k3^{k+1}} - \sum_{k=0}^{n-1} \frac{2^k - k}{k3^{k+1}} \right|$$

$$= \left| \frac{2^n - n}{n3^{n+1}} \right| < \frac{2^n}{n3^{n+1}} < \frac{2^n}{n3^n} < \frac{2^n}{3^n}$$

$$= \frac{1}{\left(\frac{3}{2}\right)^n} < \frac{1}{2^n}$$

So, the sequence is Cauchy by our earlier lemma. □

We are now finally able to give a construction of the real numbers. Let $\mathcal{C}$ be the set of all Cauchy sequences of rational numbers. We then place an equivalence relation on $\mathcal{C}$ by declaring $(x_n)_{n \in \mathbb{N}} \sim (y_n)_{n \in \mathbb{N}}$ if the limit of their differences, $\lim_{n \rightarrow \infty} (x_n - y_n)$, is zero. Write $[x_n]$ for the equivalence class containing a Cauchy sequence of rationals, x_n . We then define $\mathbb{R}$ to be the set of equivalence classes, $\mathcal{C}/\sim$. For time we won't go over the details, but to show this agrees with our earlier definition, we need to show $\mathcal{C}/\sim$ is an ordered field satisfying the completeness axiom. This requires we

- 1) Define addition and multiplication on $\mathcal{C}/\sim$
- 2) Verify that these operations satisfy the field axioms
- 3) Define a total order on $\mathcal{C}/\sim$
- 4) Verify $\mathcal{C}/\sim$ is an ordered field w/ this ordering
- 5) Show the completeness axiom holds.

Doing all of this is rather tedious, so we'll just define the operation and ordering, and leave the other parts as an (optional) exercise.

We define $[x_n] + [y_n]$ as simply the equivalence class of the sequence $(x_n + y_n)_{n \in \mathbb{N}}$ where x_n, y_n are representatives of $[x_n], [y_n]$. Note the pointwise-sum of two Cauchy sequences is Cauchy (homework problem), and this is a well-defined operation: if x'_n, y'_n are two other representatives of our equivalence classes, then $x_n - x'_n \rightarrow 0$ and $y_n - y'_n \rightarrow 0$, so

$$\begin{aligned}(x_n + y_n) - (x'_n + y'_n) &= x_n + y_n - x'_n - y'_n \\ &= x_n - x'_n + y_n - y'_n \\ &\rightarrow 0 + 0 = 0\end{aligned}$$

Thus, choosing different representatives of the equivalence class does not affect the equivalence class of $[x_n] + [y_n]$.

Multiplication is defined similarly:

$$[x_n] \cdot [y_n] = [x_n \cdot y_n]$$

and is also well-defined:

$$\begin{aligned}
 x_n y_n - x'_n y'_n &= x_n y_n - x_n y'_n + x_n y'_n - x'_n y'_n \\
 &= x_n (y_n - y'_n) + y'_n (x_n - x'_n) \\
 &\rightarrow 0 + 0 = 0.
 \end{aligned}$$

Finally, we write $[x_n] < [y_n]$ if there exists an N st. for all $n > N$, $x_n < y_n$. (Equiv., $y_n - x_n > 0$ for all $n > N$)

Finally, we end our discussion of sequences by mentioning one important property:

Thm

Let a_n, b_n be a ^{pair of} sequences of real numbers such that for every m, n we have $a_m \leq b_n$, and a_n is increasing while b_n is decreasing.

Then

$$\begin{aligned}
 \bigcap_{n=1}^{\infty} [a_n, b_n] &= [a_1, b_1] \cap [a_2, b_2] \cap [a_3, b_3] \cap \dots \\
 &\neq \emptyset.
 \end{aligned}$$

Pf

Notice every $a_n \leq b_1$, so a_n is an increasing

sequence bounded above; say $a := \lim_{n \rightarrow \infty} a_n$. Similarly, b_n is a decreasing sequence bounded below; say $b := \lim_{n \rightarrow \infty} b_n$. We claim $a \leq b$. If not, so $a > b$, then let $\varepsilon = a - b > 0$. As $a_n \rightarrow a$ and $b_n \rightarrow b$, there exists an N such that

$$|a_n - a| < \varepsilon/2, \text{ and}$$

$$|b_n - b| < \varepsilon/2$$

for all $n > N$. [There may be different N 's for the a -sequence and b -sequence, but we'll just take the maximum of these two values.] Since a_n increases and b_n decreases, $a > a_n$ and $b < b_n$. Thus when $|b_n - b| < \varepsilon/2$, we have

$$b_n - b < \varepsilon/2$$

$$\Rightarrow b_n < b + \varepsilon/2$$

Similarly, $a_n > a - \varepsilon/2$. But $a - b = \varepsilon$, so $a = b + \varepsilon$ and

$$a_n > a - \frac{\varepsilon}{2} > b + \varepsilon - \frac{\varepsilon}{2} > b + \frac{\varepsilon}{2} > b_n$$

and this contradicts our assumption each $a_n \leq b_n$.

Lecture 09 - Limits of Functions

We now turn our attention from discussing limits of sequences to limits of functions, although we'll see there is a relationship between these two types of limits.

Intuitively, we say the limit of a function $f(x)$ as x approaches a equals L if the outputs of the function get "really, really close" to L as the inputs to the function get "really, really close" to a . We make this notion precise as follows:

The limit of $f(x)$ as x approaches a is equal to L , denoted $\lim_{x \rightarrow a} f(x) = L$, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all x satisfying $0 < |x - a| < \delta$.

That is, the output $f(x)$ is guaranteed to be within ϵ -distance of L , provided the input x is within δ -distance of a — but not necessarily equal to a .

Ex

We will show $\lim_{x \rightarrow 3} (2x+7) = 13$. Let $\varepsilon > 0$ be given and take $\delta = \varepsilon/2$. Now suppose x satisfies $0 < |x-3| < \delta$. We then have

$$\begin{aligned} 0 < |x-3| < \delta \\ \Rightarrow |x-3| < \varepsilon/2 \\ \Rightarrow 2|x-3| < \varepsilon \\ \Rightarrow |2x-6| < \varepsilon \\ \Rightarrow |2x+7-13| < \varepsilon. \end{aligned}$$

Of course, the appropriate δ for this example was found by "reverse-engineering" the problem. i.e., we start from the desired conclusion, $|2x+7-13| < \varepsilon$, and work backwards to see what condition on $|x-3|$ would imply this result:

$$\begin{aligned} |2x+7-13| < \varepsilon \\ \Rightarrow |2x-6| < \varepsilon \\ \Rightarrow 2|x-3| < \varepsilon \\ \Rightarrow |x-3| < \varepsilon/2. \end{aligned}$$

For linear functions this process is easy and straight-forward, but for other functions the algebra may be more involved.

Ex

Determine the δ necessary for showing
 $\lim_{x \rightarrow 2} (3x^2 + 5) = 17$.

We again work backwards, starting from

$$\begin{aligned} & |3x^2 + 5 - 17| < \epsilon \\ \Rightarrow & |3x^2 - 12| < \epsilon \\ \Rightarrow & 3|x^2 - 4| < \epsilon \\ \Rightarrow & |x^2 - 4| < \epsilon/3 \\ \Rightarrow & |x-2||x+2| < \epsilon/3 \end{aligned}$$

Now, our goal is to isolate $|x-2|$ on one side of the inequality. Here we can not just divide by $|x+2|$ and declare $\delta = \frac{\epsilon}{3|x+2|}$.

The reason we can not do this is slightly subtle. In our definition we have an ordering of when different quantities are determined:

- We begin with an $\epsilon > 0$

- We then determine a δ . Since δ is determined after ϵ , δ may depend on ϵ .
- We then consider all x such that $0 < |x-a| < \delta$. That is, x is chosen after δ , so δ does not depend on x . But, instead, the choice of "allowable" x 's depends on δ .

So, x depends on δ , but δ does not depend on x . We care that $|x+2|$ appears on the right-hand side of the inequality above, though.

Let's notice that eventually we'll have $|x-2| < \delta$. We can use this to help us understand $|x+2|$.

$$\begin{aligned}
 &|x-2| < \delta \\
 \Rightarrow &-\delta < x-2 < \delta \\
 \Rightarrow &2-\delta < x < 2+\delta \\
 \Rightarrow &4-\delta < x+2 < 4+\delta
 \end{aligned}$$

Thus if we know $\delta \leq 1$, for example, we'd have

$$3 \leq 4-\delta < x+2 < 4+\delta \leq 5$$

So as long as we guarantee $\delta \leq 1$, we may suppose $3 < x+2 < 5$. Since these values are positive, we have $3 < |x+2| < 5$. Continuing our work from above, if $|x+2| < 5$ then

$$|x-2||x+2| < 5|x-2|$$

So it suffices to show $5|x-2| < \frac{\epsilon}{3} \Rightarrow |x-2| < \frac{\epsilon}{15}$

Thus we should take $\delta = \frac{\epsilon}{15}$, but we must also be sure $\delta \leq 1$ to use our reasoning above. We may then use $\delta = \min\{\frac{\epsilon}{15}, 1\}$. Now we proceed w/ the proof:

Let $\epsilon > 0$ be given, take $\delta = \min\{\frac{\epsilon}{15}, 1\}$. Suppose $0 < |x-2| < \delta$. Then as $\delta \leq 1$, note $3 < |x+2| < 5$ as

$$|x-2| < \delta \leq 1$$

$$\Rightarrow -1 < x-2 < 1$$

$$\Rightarrow 3 < x+2 < 5.$$

Hence

$$|x-2| < \delta$$

$$\Rightarrow |x-2||x+2| < \delta|x+2|$$

$$\Rightarrow |x^2 - 4| < \frac{\varepsilon}{15} \cdot |x+2| < \frac{\varepsilon}{15} \cdot 5 = \frac{\varepsilon}{3}$$

$$\Rightarrow 3|x^2 - 4| < \varepsilon$$

$$\Rightarrow |3x^2 - 12| < \varepsilon$$

$$\Rightarrow |3x^2 + 5 - 17| < \varepsilon$$

Hence $\lim_{x \rightarrow 2} (3x^2 + 5) = 17$.

We can relate limits of functions to limits of sequences via:

Theorem

The limit of $f(x)$ as x approaches a equals L iff for every sequence x_n contained in the domain of f and not containing a (i.e., $x_n \neq a$ for any n) such that $x_n \rightarrow a$, we have the sequence $f(x_n) \rightarrow L$.

Pf

• Suppose $\lim_{x \rightarrow a} f(x) = L$. Then $\forall \varepsilon > 0 \exists \delta > 0$ st. $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$. Let $\varepsilon > 0$ be given and suppose $x_n \rightarrow a$. Let δ be as in the def'n of $\lim_{x \rightarrow a} f(x) = L$. Let N be such that $|x_n - a| < \delta$ for all $n > N$. Then note for all $n > N$ we also have $|f(x_n) - L| < \varepsilon$.

• The other part of the iff statement is
from Gordon (Thm 3.2 on pg 85) □

Cor

If x_n, x'_n are sequences which both converge
to a , neither of which contain a and both
are in the domain of f with $f(x_n) \rightarrow L$ and
 $f(x'_n) \rightarrow M$, $L \neq M$, then $\lim_{x \rightarrow a} f(x)$ DNE. □

E.g.,

The function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

does not have a limit at any point.

Of course, you also know that in calculus
we are sometimes interested in one-sided
limits. Before giving an ϵ - δ definition for
one-sided limits, let's notice that the fact
from calc. 1 that $\lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x)$ implies
 $\lim_{x \rightarrow a} f(x)$ does not exist can be interpreted

in terms of the above corollary. Namely if x_n increases to a , $f(x_n)$ converges to the left-hand limit; and if x_n decreases to a , then $f(x_n)$ converges to the right-hand limit. Hence if these values differ, the corollary tells us the limit does not exist.

The ϵ - δ definition of the one-sided limits are:

- We write $\lim_{x \rightarrow a^+} f(x) = L$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all x satisfying $0 < x - a < \delta$.
- We write $\lim_{x \rightarrow a^-} f(x) = L$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all x satisfying $0 < a - x < \delta$.

Notice these definitions are exactly like the definition of the normal two-sided limit, but we only are concerned with $x > a$ for the right-hand limit, and $x < a$ for the left-hand limit.

E.g.,

Show $\lim_{x \rightarrow 3^+} \sqrt{x-3} = 0$.

We again "reverse-engineer" the problem:

$$\begin{aligned} & |\sqrt{x-3} - 0| < \varepsilon \\ \Rightarrow & \sqrt{x-3} < \varepsilon \\ \Rightarrow & x-3 < \varepsilon^2 \end{aligned}$$

So, our proof works as follows:

Let $\varepsilon > 0$ be given and let $\delta = \varepsilon^2$.

If $0 < x-3 < \delta$, then

$$\begin{aligned} & x-3 < \delta \\ \Rightarrow & x-3 < \varepsilon^2 \\ \Rightarrow & \sqrt{x-3} < \varepsilon \\ \Rightarrow & |\sqrt{x-3} - 0| < \varepsilon \end{aligned}$$

Just as we can interpret the normal, two-sided limit in terms of sequences, we can do something similar for one-sided limits.

Thm

• The right-hand limit of $f(x)$ as

x approaches a from the right exists and equals L if and only if for all sequences x_n which decrease to a (and where each x_n is in the domain of $f(x)$ and no x_n equals a), we have $f(x_n) \rightarrow L$.

• The left-hand limit of $f(x)$ as x approaches a from the left exists and equals L if and only if for all sequences x_n which increase to a (and where each x_n is in the domain of $f(x)$ and no x_n equals a), we have $f(x_n) \rightarrow L$.

Pf
Exercise.

Sometimes working directly with the ϵ - δ definitions is difficult, or at least tedious, so we may wish to have some theorems that can help us side-step the definition. The following is an example of such a theorem.

Thm (Sandwich/Squeeze Theorem)

Suppose $f(x), g(x), h(x)$ are three functions such that $f(x) \leq g(x) \leq h(x)$ for all x near a .

(i.e., there exists some interval (α, β) containing a where the inequalities hold.) If $\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} h(x)$, then $\lim_{x \rightarrow a} g(x) = L$.

Pf

Let x_n be any sequence such that $x_n \rightarrow a$. Then $f(x_n)$ and $h(x_n)$ converge to L , and for all sufficiently large n , $f(x_n) \leq g(x_n) \leq h(x_n)$. Hence by the sandwich theorem for sequences, $g(x_n) \rightarrow L$ as well. The sequence $x_n \rightarrow a$ was arbitrary, so $g(x_n) \rightarrow L$ for all sequences x_n converging to a , so $\lim_{x \rightarrow a} g(x) = L$. $\square$

E.g., Show $\lim_{x \rightarrow 0} \frac{x^4}{x^4 + x^2 + 1} = 0$.

Notice

$$x^2 < x^4 + x^2 + 1$$

$$\Rightarrow \frac{x^2}{x^4 + x^2 + 1} < 1$$

$$\Rightarrow \frac{x^2}{x^4 + x^2 + 1} \cdot x^2 < 1 \cdot x^2$$

$$\Rightarrow \frac{x^4}{x^4 + x^2 + 1} < x^2$$

Also note $0 \leq \frac{x^4}{x^4 + x^2 + 1}$, so $0 \leq \frac{x^4}{x^4 + x^2 + 1} \leq x^2$

Since $\lim_{x \rightarrow 0} x^2 = 0$ [Let $\varepsilon > 0$ be given, take $\delta = \sqrt{\varepsilon}$, so $0 < x < \delta \Rightarrow x < \sqrt{\varepsilon} \Rightarrow x^2 = |x^2| < \varepsilon$], the sandwich theorem tells us $\lim_{x \rightarrow 0} \frac{x^4}{x^4 + x^2 + 1} = 0$.

We end our discussion of limits of function by defining what it means to say $\lim_{x \rightarrow a} f(x) = \infty$.

We write $\lim_{x \rightarrow a} f(x) = \infty$ if for every $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ for all x satisfying $0 < |x - a| < \delta$.

E.g.

Show $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$.

We work backwards to find δ :

$$\begin{aligned} \frac{1}{x^2} &> M \\ \Rightarrow 1 &> Mx^2 \\ \Rightarrow \frac{1}{M} &> x^2 \\ \Rightarrow x &< \frac{1}{\sqrt{M}} \end{aligned}$$

Question for students:
what if $M < 0$? Then $\frac{1}{M} > x^2$ is false.
Why is this not a problem?

Let M be given. If $M \leq 0$, then take $\delta = 1$
and note for $0 < |x - 0| < \delta$, $\frac{1}{x^2} > 0 \geq M$.

If $M > 0$, take $\delta = \frac{1}{\sqrt{M}}$. Then for all x satisfying $0 < |x - 0| < \delta$ we have

$$\begin{aligned} & |x| < \delta \\ \Rightarrow & |x| < \frac{1}{\sqrt{M}} \\ \Rightarrow & \sqrt{M} < \frac{1}{|x|} \\ \Rightarrow & (\sqrt{M})^2 < \left(\frac{1}{|x|}\right)^2 \\ \Rightarrow & M < \frac{1}{x^2}. \end{aligned}$$

Lecture 10 - Continuity

At the start of the semester we said that real analysis was the study of "nice" functions of a real variable. Today we introduce our first "nice" property, which is continuity.

The intuitive behind behind continuity is simply that "small" changes to the input of the function should result in "small" changes to the output. I.e., the function does not instantaneously change from one value to the next, but must change gradually. (This is codified by the intermediate value theorem which we'll see shortly.) Put another way, if $f(a) = L$, then x -values "near" a should produce $f(x)$ values "near" L . We make this idea precise as follows:

A function $f(x)$ is said to be continuous at a if a is in the domain of f , and for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ for all x satisfying $|x - a| < \delta$.

Notice this is really just saying $\lim_{x \rightarrow a} f(x) = f(a)$.
Thus we could alternatively define continuity of f at a as the three following conditions being satisfied:

- a is in the domain of f ,
- $\lim_{x \rightarrow a} f(x)$ exists, and
- $\lim_{x \rightarrow a} f(x) = f(a)$.

Ex

- If $f(x) = k$ is a constant function, then $f(x)$ is continuous at every value of a :

Let $\epsilon > 0$ be given. Let $\delta = 1$. Note for all x satisfying $|x - a| < \delta$ we have $f(x) = k = f(a)$, so $|f(x) - f(a)| = 0 < \epsilon$.

- If $f(x) = x$ is the identity function, then $f(x)$ is continuous at every value of a :

Let $\epsilon > 0$ be given, and take $\delta = \epsilon$. Then if $|x - a| < \delta$ we have

$$|f(x) - f(a)| = |x - a| < \delta = \epsilon.$$

Though these two functions above seem trivial, they will turn out to be invaluable: as we

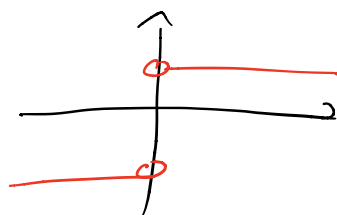
will see, these two simple functions will form building blocks for much more interesting functions.

Notice a function may fail to be continuous at a point in three different ways:

- the function may not be defined at a ;
- the limit of the function may not exist;
- the value of the function may not equal the value of the limit.

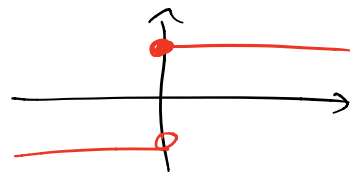
The functions $f_1(x)$, $f_2(x)$, and $f_3(x)$ below exhibit each of these failures:

$$f_1(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$



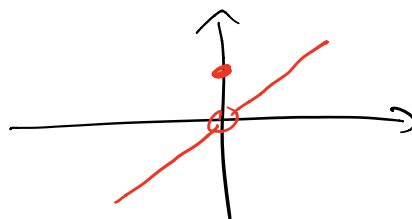
is undefined at 0, so not continuous at zero.

$$f_2(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$



is defined at 0, but the limit does not exist

$$f_3(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$



is defined at 0 and the limit exists, but these values are not equal.

When you first learn about continuity in calculus, most of the discontinuous functions you encounter are continuous everywhere except at a few isolated points, but more exotic things can happen.

For instance, the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is not continuous at any point since the limit $\lim_{x \rightarrow a} f(x)$ does not exist for any choice of a .

The function

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is continuous at 0, but nowhere else:

- Continuity at 0

Let $\varepsilon > 0$ be given and let $\delta = \varepsilon$.

Note if $|x - 0| < \delta$, then

$$|f(x) - 0| = |f(x)| = |\pm x| = |x| < \delta = \varepsilon.$$

- Continuity nowhere else

Let $a \neq 0$. Let x_n be a sequence of rational numbers converging to a , and x'_n a sequence of irrationals converging to a .

Then $f(x_n) = x_n \rightarrow a$, but $f(x'_n) = -x'_n \rightarrow -a$.

Hence $\lim_{x \rightarrow a} f(x)$ DNE.

If a function is continuous at every point in its domain, then we simply say the function is continuous. Given an interval $I \subseteq \mathbb{R}$, we say $f(x)$ is continuous on I if it is continuous at every point of I .

- $\frac{1}{x}$ is not continuous at 0
- $\frac{1}{x}$ is continuous on $(0, 1)$
- $\frac{1}{x}$ is not continuous on $[-1, 1]$
- $\frac{1}{x}$ is continuous

← Notice $f(x) = k$
and $f(x) = x$ are
continuous

One of the nice things about continuous functions is that we can combine them together in various ways to get new continuous functions

Thm

Suppose $f(x)$ and $g(x)$ are continuous at a .

Then so are $f(x) \pm g(x)$, $f(x)g(x)$, and $\frac{f(x)}{g(x)}$ (provided $g(a) \neq 0$).

Pf

Let x_n be any sequence converging to a .
Then $f(x_n) \rightarrow f(a)$ and $g(x_n) \rightarrow g(a)$, so by our theorems of limits of sequences,

$$\bullet f(x_n) + g(x_n) \rightarrow f(a) + g(a)$$

$$\bullet f(x_n) - g(x_n) \rightarrow f(a) - g(a)$$

$$\bullet f(x_n)g(x_n) \rightarrow f(a)g(a)$$

$$\bullet \frac{f(x_n)}{g(x_n)} \rightarrow \frac{f(a)}{g(a)} \quad (\text{provided no } x_n = a, \text{ and no } g(x_n) = 0) \quad \square$$

Cor

If $f(x)$, $g(x)$ are continuous on I , or simply continuous everywhere in their domains, so are $f(x) \pm g(x)$, $f(x)g(x)$, $\frac{f(x)}{g(x)}$. $\square$

As a consequence, notice that polynomials and rational functions are continuous. Proving other functions we know and love are continuous is a bit trickier.

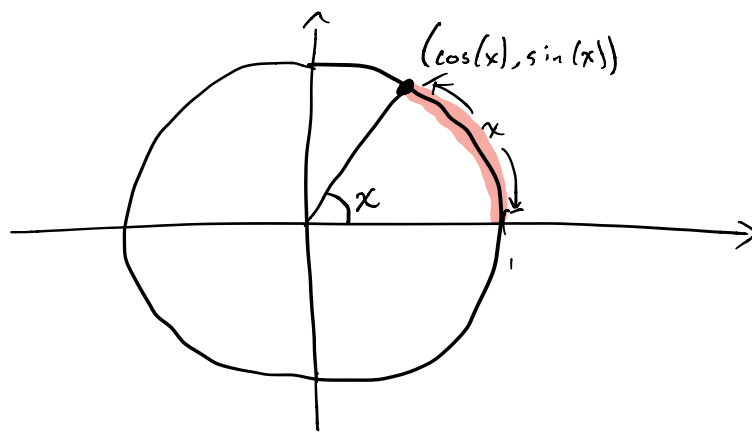
Thm

$\sin(x)$ is continuous

Pf

We need to first assemble a few facts about $\sin(x)$.

Note for every x we have $|\sin(x)| \leq |x|$. This follows from the geometric definition of $\sin(x)$:



In particular as the highlighted arc of the circle above has length x , we see $\sin(x) \leq x$ for all $x \in [0, \frac{\pi}{2}]$. Since $\frac{\pi}{2} > 1$ and $\sin(x) \leq 1$ for all x , this shows $\sin(x) \leq x$ for all $x \geq 0$.

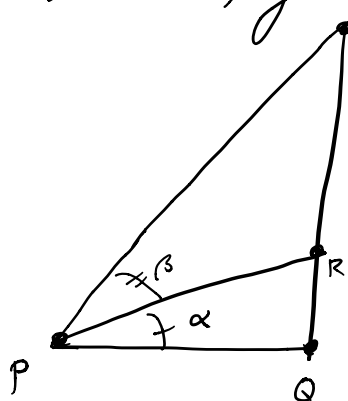
By symmetry, $|\sin(x)| \leq |x|$ for $x \in [-\frac{\pi}{2}, 0]$ and since $-\frac{\pi}{2} < -1$ but $\sin(x) \geq -1$ for all x , we

have $|\sin(x)| \leq |x|$ for all $x < 0$. Together with $\sin(x) \leq x$ for all $x > 0$ this establishes $|\sin(x)| \leq |x|$ for all x .

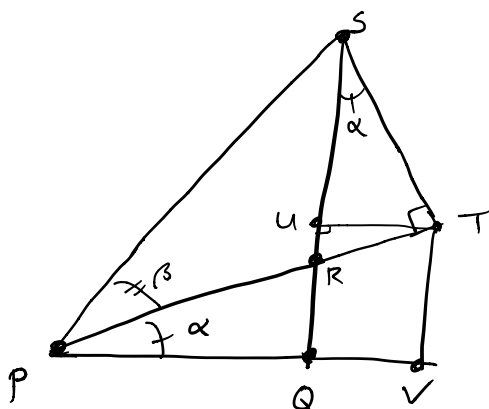
Now recall the angle sum and difference formulas for $\sin(x)$. First consider the sum identity

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$

We prove this by considering



Then $\sin(\alpha + \beta) = \frac{|QS|}{|PS|}$. We compute this by amending our figure:



Now note $|QS| = |QU| + |US| = |VT| + |US|$, so

$$\begin{aligned}\sin(\alpha + \beta) &= \frac{|VT|}{|PS|} + \frac{|US|}{|PS|} \\&= \frac{|VT|}{|PS|} \cdot \frac{|PT|}{|PT|} + \frac{|US|}{|PS|} \cdot \frac{|ST|}{|ST|} \\&= \frac{|VT|}{|PT|} \cdot \frac{|PT|}{|PS|} + \frac{|US|}{|ST|} \cdot \frac{|ST|}{|PS|} \\&= \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)\end{aligned}$$

Now note

$$\begin{aligned}\sin(\alpha - \beta) &= \sin(\alpha + (-\beta)) \\&= \sin(\alpha) \cos(-\beta) + \sin(-\beta) \cos(\alpha)\end{aligned}$$

since $\cos(x)$ is an even function, $\cos(-\beta) = \cos(\beta)$; but $\sin(x)$ is an odd function, so $\sin(-\beta) = -\sin(\beta)$, and the above becomes

$$\sin(\alpha - \beta) = \sin(\alpha) \cos(\beta) - \sin(\beta) \cos(\alpha)$$

Now note for any x, y we may write

$$x = \frac{x+y}{2} + \frac{x-y}{2}, \quad y = \frac{x+y}{2} - \frac{x-y}{2}$$

and using the identities above may write

$$\begin{aligned}\sin(x) - \sin(y) &= \sin\left(\frac{x+y}{2} + \frac{x-y}{2}\right) - \sin\left(\frac{x+y}{2} - \frac{x-y}{2}\right) \\&= \sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) + \sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right) - \left[\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) - \sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)\right] \\&= \cancel{\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)} + \sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right) - \cancel{\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)} + \sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right) \\&= 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)\end{aligned}$$

Finally, we'll put all of this together to show $\sin(x)$ is continuous at every real number.

Let $a \in \mathbb{R}$ be any real number and let $\varepsilon > 0$ be given. We let $\delta = \varepsilon$ and note if $|x - a| < \delta$, then

$$\begin{aligned}|\sin(x) - \sin(a)| &= \left| 2\sin\left(\frac{x-a}{2}\right)\cos\left(\frac{x+a}{2}\right) \right| \\&\leq 2\left|\sin\left(\frac{x-a}{2}\right)\right| \quad \left(\text{as } -1 \leq \cos\left(\frac{x+a}{2}\right) \leq 1\right) \\&\leq 2 \cdot \left|\frac{x-a}{2}\right| \quad \left(\text{this is } |\sin(x)| \leq |x| \text{ for all } x\right) \\&= |x-a| < \delta = \varepsilon\end{aligned}$$

□

Lecture 11 - Consequences of Continuity; Basic Topological Ideas

Once we know a function $f(x)$ is continuous, several nice properties immediately follow. Here we'll mention two particular properties, both of which you saw in first semester calculus, and both of which require the completeness axiom.

The first property, the Intermediate Value Theorem, essentially tells us that continuous functions can't "skip" over values (at least, as long as their domain is an interval).

Thm (Intermediate Value Theorem)

Suppose $f(x)$ is continuous on an interval $[a, b]$ where $f(a) = A$ and $f(b) = B$.

Then for any $C \in (\min(A, B), \max(A, B))$, there is some $c \in (a, b)$ so that $f(c) = C$.

Pf

Suppose $A < B$, and define $S = \{x \in (a, b) \mid f(x) \leq C\}$.
As $A < C < B$, there exists some $\varepsilon > 0$ such that $A + \varepsilon < C$ and $B - \varepsilon > 0$. Since $\lim_{x \rightarrow a^+} f(x) = A$ and $\lim_{x \rightarrow b^-} f(x) = B$, there is a $\delta > 0$ such that

$$0 < x - a < \delta \implies |f(x) - A| < \varepsilon$$

$$0 < b - x < \delta \implies |f(x) - B| < \varepsilon$$

This means S is not empty (it contains all x in $(a, a+\delta)$ and is bounded above (by every x in $(b-\delta, b)$). By the completeness axiom, the supremum of S exists. Define $c := \sup S$. We need to show $f(c) = C$. Letting c_n be a sequence in S converging to c , continuity tells us $f(c_n) \rightarrow f(c)$. As every $f(c_n) \leq C$, $f(c) \leq C$ too. Define $d_n = c + \frac{b-c}{n+1}$. Note each $d_n \in (a, b)$ and d_n is not in S since each $d_n > c$, but $c = \sup S$. By the definition of S , this means $f(d_n) > C$. Since d_n converges to c and f is continuous this means $f(c) \geq C$.

(The proof when $A > B$ is similar, and the statement is trivial if $A = B$.) □

The other main property we will prove is that if a function is continuous on a closed, bounded interval, then it has a global maximum and minimum on that interval.

Here we will deviate from Gordon by introducing

some topological ideas. Doing this will allow us to give proofs that hold more generally, and after introducing the basic language and ideas, some proofs will also become simpler.

To begin, we will define an ε -ball around a point $x \in \mathbb{R}$, for $\varepsilon > 0$, as the set

$$B_\varepsilon(x) := \{y \in \mathbb{R} \mid |x - y| < \varepsilon\}$$

That is, $B_\varepsilon(x)$ consists of the points which are ε -close to x . Notice $B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$.

We will say a set $U \subseteq \mathbb{R}$ is open if for every point $x \in U$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

For example, every open interval is open.

- If $U = (a, b)$, for $x \in U$ consider $B_\varepsilon(x)$ where $\varepsilon = \frac{1}{2} \min\{|b - x|, |x - a|\}$
- If $U = (a, \infty)$, take $\varepsilon = \frac{1}{2}|x - a|$
- If $U = (-\infty, b)$, take $\varepsilon = \frac{1}{2}|b - x|$
- If $U = (-\infty, \infty)$, any $\varepsilon > 0$ suffices.

The empty set is vacuously open.

Thm

Any union of open sets is open

Pf

Let $U_i, i \in I$, be any collection of open subsets of $\mathbb{R}$, and consider $U = \bigcup_{i \in I} U_i$. Then for any $x \in U$, $x \in U_i$ for some U_i . Hence by assumption $\exists \varepsilon > 0$ st. $B_\varepsilon(x) \subseteq U_i$, so $B_\varepsilon(x) \subseteq U$. □

So, sets which are built from unioning up open sets, like $(-\infty, 0) \cup (0, \infty)$ or $(-2, 3) \cup (7, 9) \cup (13, \infty)$ are also open. In fact, this is the only way we can obtain open sets.

Thm

Every open set in $\mathbb{R}$ is a union of open intervals

Pf

If U is an open set, then by definition for each $x \in U$ $\exists \varepsilon > 0$ st. $B_\varepsilon(x) \subseteq U$. This ε changes depending on the $x \in U$ under

consideration, so we should think of ε as a function of x , $\varepsilon(x)$. Then we claim

$$\mathcal{U} = \bigcup_{x \in \mathcal{U}} B_{\varepsilon(x)}(x)$$

- $\bigcup_{x \in \mathcal{U}} B_{\varepsilon(x)}(x) \subseteq \mathcal{U}$ is clear as each $B_{\varepsilon(x)}(x) \subseteq \mathcal{U}$.
- $\mathcal{U} \subseteq \bigcup_{x \in \mathcal{U}} B_{\varepsilon(x)}(x)$ as for each $x_0 \in \mathcal{U}$ we

have $x_0 \in B_{\varepsilon(x_0)}(x_0)$, so $x_0 \in \bigcup_{x \in \mathcal{U}} B_{\varepsilon(x)}(x)$. □

A closed subset of $\mathbb{R}$ is the complement of an open subset. Notice every closed interval is a closed set:

$$[a, b] = ((-\infty, a) \cup (b, \infty))^c$$

$$= \mathbb{R} \setminus ((-\infty, a) \cup (b, \infty))$$

$$[a, \infty) = (-\infty, a)^c$$

$$(-\infty, b] = (b, \infty)^c$$

$$\{a\} = [a, a] = ((-\infty, a) \cup (a, \infty))^c$$

Notice that since $\emptyset$ is open, its complement, $\mathbb{R} = (-\infty, \infty)$, is closed. Similarly, $(-\infty, \infty)$ is open so $\emptyset$ is closed. That is, $\mathbb{R}$ and $\emptyset$ are both open and closed. Sometimes such sets are called clopen.

Thm

Every intersection of closed sets is closed.

Pf

Exercise — use De Morgan's laws. □

So, open sets are preserved under all unions (finite, countable, and uncountable), and closed sets are similarly preserved under all intersection. But what about intersections of open sets and unions of closed sets?

It is possible to take an intersection of open sets and get a closed set:

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right) = [0, 1]$$

However, this does require we take an intersection of infinitely-many open sets.

Thm

A finite intersection of open sets is open.

Pf

$U = U_1 \cap \dots \cap U_n$. If $x \in U$, then

$x \in U_i$, $i = 1, \dots, n$. Hence $\exists \varepsilon_i$ st. $B_{\varepsilon_i}(x) \subseteq U_i$

Let $\varepsilon = \min \{ \varepsilon_1, \dots, \varepsilon_n \}$. Note $\varepsilon \leq \varepsilon_i$ for each

i , so $B_\varepsilon(x) \subseteq B_{\varepsilon_i}(x)$ for each i , hence

$B_\varepsilon(x) \subseteq U_i$ for each i , so $B_\varepsilon(x) \subseteq \bigcap_{i=1}^n U_i$ □

Cor

A finite union of closed sets is closed.

Pf

Exercise - use De Morgan's laws. □

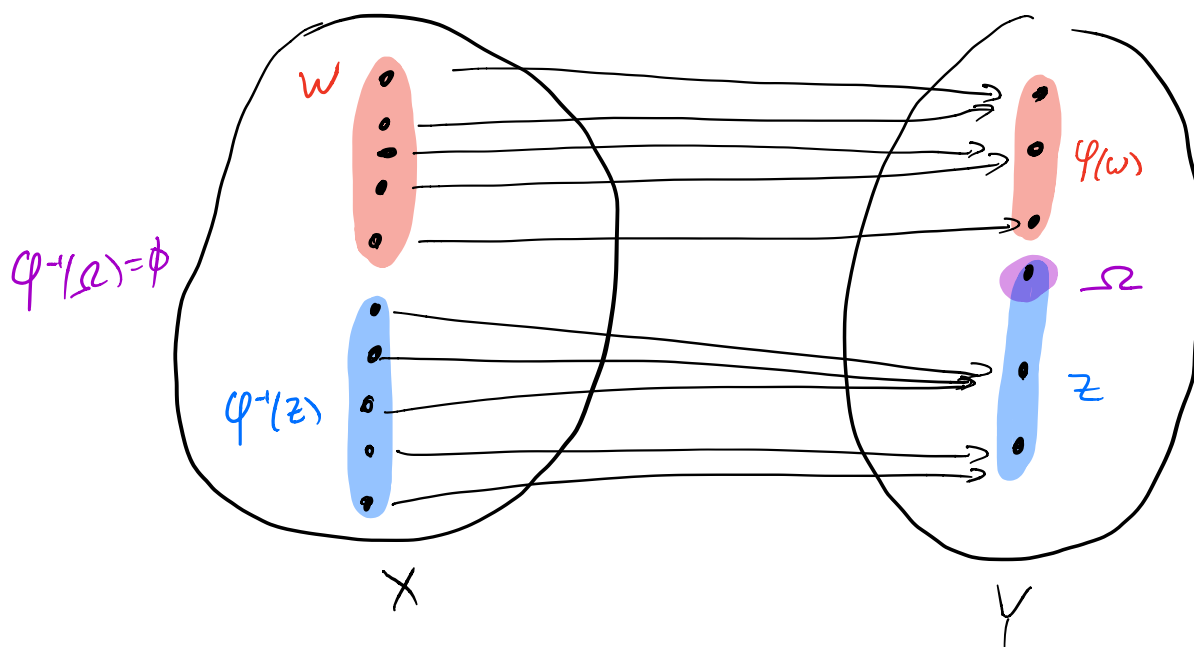
Open sets are closely related to the idea of continuity. To see this we need to recall some basic facts about preimages.

Recall that if $\varphi: X \rightarrow Y$ is a map between sets and $W \subseteq X$, then the image of W under φ is

$$\varphi(W) := \{ \varphi(w) \mid w \in W \}$$

If $Z \subseteq Y$, the preimage of Z under φ is

$$\varphi^{-1}(Z) := \{x \in X \mid \varphi(x) \in Z\}$$



Notice preimages preserve unions: $\varphi^{-1}(Z_1 \cup Z_2) = \varphi^{-1}(Z_1) \cup \varphi^{-1}(Z_2)$:

$$\begin{aligned} \varphi^{-1}(Z_1 \cup Z_2) &= \{x \in X \mid \varphi(x) \in Z_1 \cup Z_2\} \\ &= \{x \in X \mid \varphi(x) \in Z_1\} \cup \{x \in X \mid \varphi(x) \in Z_2\} \\ &= \varphi^{-1}(Z_1) \cup \varphi^{-1}(Z_2) \end{aligned}$$

Thm

A function $f: X \xrightarrow{\subseteq \mathbb{R}} \mathbb{R}$ is continuous iff the preimage of every open set is open.

Pf

- Suppose for every open $U \subseteq \mathbb{R}$, $f^{-1}(U)$ is open. We must show for every $a \in X$ and every $\varepsilon > 0$, $\exists \delta$ st. $|f(x) - f(a)| < \varepsilon$ for all x satisfying $|x - a| < \delta$. Notice that $B_\varepsilon(f(a))$ is an open subset of $\mathbb{R}$. By assumption, its preimage $f^{-1}(B_\varepsilon(f(a)))$ is open. Note a is an element of this preimage, hence there is some $\delta > 0$ st. $B_\delta(a)$ is contained in the preimage. That is $f(B_\delta(a)) \subseteq B_\varepsilon(f(a))$. But this exactly means if $x \in (a - \delta, a + \delta)$, then $f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$ — i.e., $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$.

- Now suppose f is continuous at every $a \in X$. Let $U \subseteq \mathbb{R}$ be open. We want to show $f^{-1}(U)$ is open. Let $a \in f^{-1}(U)$, so $f(a) \in U$. Hence for some $\varepsilon > 0$, $B_\varepsilon(f(a)) \subseteq U$. Since f is continuous at a , $\exists \delta$ st. $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$. This exactly means $f(B_\delta(a)) \subseteq B_\varepsilon(f(a)) \subseteq U$. Thus $B_\delta(a) \subseteq f^{-1}(U)$. This is true for all $a \in f^{-1}(U)$, so $f^{-1}(U)$ is open. $\square$

Since the above theorem was "iff," we can use this as the definition of a continuous map: $f: \mathbb{R} \rightarrow \mathbb{R}$

is continuous if for every open $U \subseteq \mathbb{R}$, the preimage $f^{-1}(U)$ is open.

One of the nice properties of continuous maps is that they will be guaranteed to have maxima and minima if we restrict them to a certain type of domain.

We say a subset $K \subseteq \mathbb{R}$ is compact if every sequence x_n which only contains elements of K has a subsequence which converges to an element of K . The Bolzano-Weierstrass theorem told us that if a sequence is bounded it has a convergent subsequence, so it shouldn't be too surprising that boundedness is a necessary condition for compactness. To be more precise, we should carefully define what it means for a general subset $S \subseteq \mathbb{R}$ to be bounded.

We say $S \subseteq \mathbb{R}$ is bounded if there exists an $M \in \mathbb{R}$ such that $|s| \leq M$ for all $s \in S$. If S is not bounded, then for every $M \in \mathbb{R}$, there exists an $s \in S$ with $|s| > M$.

E.g., $[0, 1)$, $[-5, 72]$, $(-\pi, e^3)$ are all bounded sets; but $(-\infty, 0)$, $\mathbb{N}$, and $\{\sqrt{n} \mid n \in \mathbb{N}\}$ are all unbounded.

Lemma

If K is compact, then K is bounded.

Pf

We'll prove the contrapositive: if K is not bounded, then K is not compact - i.e., there is a sequence in K which does not converge. So suppose $K \subseteq \mathbb{R}$ is not bounded. Then for every $n \in \mathbb{N}$ there exists an $x_n \in K$ st. $|x_n| > n$. This gives an unbounded sequence of elements in K , so x_n can not converge. $\square$

Being bounded is not quite "good enough" to be compact, as the next example shows: the set $(0, 1)$ is bounded and the sequence $x_n = \frac{1}{n}$ is contained in the set. While $\frac{1}{n} = x_n$ converges, it doesn't converge to an element of $(0, 1)$.

To "fix" this we want to consider what elements of $\mathbb{R}$ can be obtained as limits of sequences in $S \subseteq \mathbb{R}$.

We call a point $\alpha \in \mathbb{R}$ an accumulation point (or a limit point) of a set S if there is a sequence s_n contained in S with $s_n \rightarrow \alpha$. Note these accumulation points need not be elements of S in general.

Prop

A set $S \subseteq \mathbb{R}$ is closed if and only if it contains all of its accumulation points.

Pf

- Suppose S is closed, so its complement S^c is open. We need to show S^c contains no accumulation point of S . If α is an accumulation pt of S , say $s_n \rightarrow \alpha$ w/ every $s_n \in S$, then $\forall \epsilon > 0$ there exist s_n st. $|\alpha - s_n| < \epsilon$. This means every ϵ -ball around α contains elements of S , so α can not be in S^c .
- Suppose S contains all its accumulation points. If $x \notin S$, then there's an $\epsilon > 0$ st. no element $s \in S$ satisfies $|x - s| < \epsilon$. Otherwise for every $\epsilon > 0$ $\exists s \in S$ w/ $|x - s| < \epsilon$. Then we could construct a seq. of elts of S — say $s_n \in S$ st. $|x - s_n| < \frac{1}{n}$ — converging to x .

But then x would be an accumulation point of S and we have a contradiction since $x \in S^c$ and S contains all of its accumulation points. Thus there is an ε -ball around each $x \in S^c$ contained in S^c , so S^c is open. $\square$

Notice that a compact set K must contain all of its accumulation points. Otherwise a sequence contained in K could converge to a point outside of K , and so all subsequences ^{of that sequence} converge to the point not in K , contradicting that K is compact. Thus we have

Lemma

A compact set must be closed.

Of course, combined with our earlier lemma, this means a compact set must be closed and bounded. Moreover we have

Thm (Heine-Borel Theorem)

A set $K \subseteq \mathbb{R}$ is compact iff it is closed and bounded.

Pf

Above we've shown that compact sets are closed and bounded, so now we show the converse.

Suppose $K \subseteq \mathbb{R}$ is closed and bounded. Let x_n be a sequence contained in K . Since K is bounded, the Bolzano-Weierstrass theorem tells us x_n has a convergent subsequence, x_{n_i} . Since each $x_n \in K$, this ^{convergent} subsequence is contained in K . As K is closed it contains all of its accumulation points, so x_{n_i} must converge to an element of K . □

The Heine-Borel theorem thus tells us closed bounded intervals must be compact.

Notice that open and closed sets aren't "nicely behaved" under continuous maps: the image of an open set may be closed, and the image of a closed set may be open:

- The image of the open set $\mathbb{R}$ under the continuous map $\sin(x)$ is the closed set $[-1, 1]$.
- The image of the closed set $\mathbb{R}$ under the continuous map $\tan^{-1}(x)$ is the open set $(-\frac{\pi}{2}, \frac{\pi}{2})$.

However, compact sets are nicely behaved:

Thm

The image of a compact set under a continuous map is compact.

Pf

Let K be a compact set and $f: K \rightarrow \mathbb{R}$ continuous. Let y_n be any sequence in $f(K)$. We must show y_n has a convergent subsequence whose limit is in $f(K)$. Let $x_n \in K$ be an elt. such that $f(x_n) = y_n$.

Since K is compact, there is some convergent subsequence x_{n_i} contained in K . Say $z \in K$ is the limit of x_{n_i} . By continuity,
$$y_{n_i} = f(x_{n_i}) \rightarrow f(z) \in f(K).$$

□

It will turn out that intervals are also nicely behaved under continuous maps.

Lemma

If $I \subseteq \mathbb{R}$ is an interval and $f: I \rightarrow \mathbb{R}$ is continuous, then $f(I)$ is also an interval.

Pf

Homework.

As a corollary, we have that the image of a closed, bounded interval under a continuous map is also a closed, bounded interval, and this implies the following important result.

Thm (Extreme Value Theorem)

A continuous function f defined on a closed, bounded interval I has a (global) maximum and a (global) minimum.

Pf
The image $f(I)$ is a closed bounded interval, say $[m, M]$. Thus $m \leq f(x) \leq M$ for all x . Furthermore, $\exists x_m, x_M \in I$ st. $f(x_m) = m$, $f(x_M) = M$. □

At this point we are only saying that a continuous function on a closed, bounded interval (or, more generally, a compact domain) must have a global max and a global min, and we are not prescribing a method for computing those values.

Lecture 12 - Uniform Continuity

We now briefly turn our attention to a special family of continuous functions called "uniformly continuous." To motivate why this will be a useful and interesting property, let's consider the following problem: Suppose $S \subseteq \mathbb{R}$ and $f: S \rightarrow \mathbb{R}$ is continuous. Does there necessarily exist a continuous function $\tilde{f}: \mathbb{R} \rightarrow \mathbb{R}$ so that $\tilde{f}|_S = f$? (We call $\tilde{f}$ an extension of f .)

There are some easy examples when such an $\tilde{f}$ does exist, such as if $S = (0, 1)$ and $f: (0, 1) \rightarrow \mathbb{R}$ is simply $f(x) = x^2$; we could define $\tilde{f}(x) = x^2$ or

$$\tilde{f}(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 > x \end{cases}$$

or several other possible extensions $\tilde{f}$.

There are also functions for which no extension exists, such as $S = (0, 1)$ and $f: S \rightarrow \mathbb{R}$ being $f(x) = \frac{1}{x}$. There is no way to extend this to a continuous function $\tilde{f}$ defined at 0. (Exercise)

So, a better question is what conditions on a continuous function $f: S \rightarrow \mathbb{R}$ will ensure the existence of a continuous extension $\tilde{f}: \mathbb{R} \rightarrow \mathbb{R}$?

One condition depends on the set S :

Thm (Tietze extension theorem)

If S is a closed set, then for every continuous $f: S \rightarrow \mathbb{R}$, there exists a continuous extension $\tilde{f}: \mathbb{R} \rightarrow \mathbb{R}$.

Pf

One "constructive" proof will be outlined in the homework. We will also be able to establish this result as a corollary of some later work. □

The Tietze extension theorem placed a condition on the domain of f ; are there conditions independent of the domain, though?

To answer this, let's remark that in the ϵ - δ definition of continuity at a point,

$f(x)$ is continuous at a if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x-a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

The δ described may depend on both the given ϵ and the value of a . For example, if we were to show the function $\frac{1}{x}$ was continuous at a point $a > 0$, we'd do something like the following: for a given $\epsilon > 0$ we wish to exhibit a δ so that $|x-a| < \delta$ implies $|\frac{1}{x} - \frac{1}{a}| < \epsilon$. We would "reverse engineer" the problem by starting with our desired conclusion, $|\frac{1}{x} - \frac{1}{a}| < \epsilon$, and working backwards to find the necessary δ . This would lead us to the following

$$|\frac{1}{x} - \frac{1}{a}| < \epsilon$$

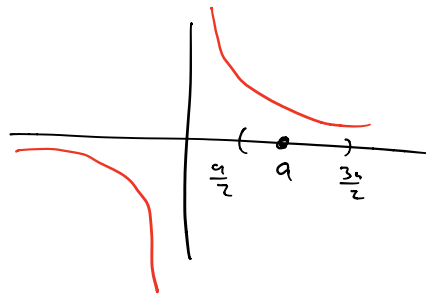
$$\Rightarrow \left| \frac{a-x}{ax} \right| < \epsilon$$

$$\Rightarrow \frac{|x-a|}{|ax|} < \epsilon$$

Supposing δ must be less than $\frac{a}{2}$, we could

use

$$\begin{aligned} |x-a| &< \frac{a}{2} \\ -\frac{a}{2} &< x-a < \frac{a}{2} \\ \frac{a}{2} &< x < \frac{3a}{2} \end{aligned}$$



Now observe

$$\frac{|x-a|}{|ax|} = \frac{|x-a|}{ax} < \frac{|x-a|}{a \cdot \frac{a}{2}} = \frac{2|x-a|}{a^2}$$

so it suffices to make $\frac{2|x-a|}{a^2}$ less than ϵ ; i.e.,

$$|x-a| < \frac{\epsilon a^2}{2}$$

So we'd take $\delta = \frac{\epsilon a^2}{2}$. Notice this depends on the point a under consideration.

Intuitively, the function $\frac{1}{x}$ grows so quickly near zero, and very slowly far away from 0, that we should expect δ to be very small for $a \approx 0$ and very large for $a \gg 0$.

It's this fact that δ depends on a that's at the root of our trouble if we try

to extend a continuous function, so we'd like to avoid this when possible, and when it is possible we say our function is "uniformly continuous."

More precisely, we say a function $f: S \rightarrow \mathbb{R}$ is uniformly continuous on S if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in S$ we have that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Notice that this means f is continuous at every point in S , but moreover the δ 's depend only on ϵ , and not on our points.

As we've seen above, $\frac{1}{x}$ is not uniformly continuous on $(0, \infty)$. It's worth pointing out, though, that $\frac{1}{x}$ is uniformly continuous on any (a, b) not containing 0. (Exercise)

As an example of another function which is uniformly continuous, consider $f(x) = \sin(x)$ defined on the entire real line, as can be seen from our earlier proof that $\sin(x)$ was continuous at every point — there we had that $\delta = \epsilon$ for each

point.

As we will see, uniformly continuous functions always have continuous extensions.

Thm

If $f: S \rightarrow \mathbb{R}$ is uniformly continuous on S , then there exists a continuous $\tilde{f}: \mathbb{R} \rightarrow \mathbb{R}$ such that $\tilde{f}|_S = f$.

Pf

For $x \in S$, we define $\tilde{f}(x) = f(x)$. We must define $\tilde{f}(x)$ for $x \notin S$. Let $x \in S^c$. We consider two cases: if x is an accumulation point of S or not.

• Suppose x is an accumulation point of S and let x_n be any sequence in S converging to x . We want to show $\lim_{n \rightarrow \infty} f(x_n)$ exists and define $\tilde{f}(x)$ to be this value. To see existence of the limit, we'll show $f(x_n)$ is Cauchy. Let $\varepsilon > 0$ be given. Since f is uniformly continuous, there is a $\delta > 0$ so that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Since x_n is a convergent sequence, it is Cauchy, so there exists an N st. if $m, n > N$, then $|x_m - x_n| < \delta$. Thus $|f(x_m) - f(x_n)| < \varepsilon$ for all $m, n > N$, so $f(x_n)$ is Cauchy and the limit $\lim_{n \rightarrow \infty} f(x_n)$ exists. Notice the value of this limit is independent of the sequence $x_n \rightarrow x$ we chose. If x'_n is another sequence in S with $x'_n \rightarrow x$, then $|x_n - x'_n| \rightarrow 0$ so there exists an N making $|x_n - x'_n| < \delta$ for $n > N$, which means $|f(x_n) - f(x'_n)| < \varepsilon$, so $|f(x_n) - f(x'_n)| \rightarrow 0$, and $\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} f(x'_n)$. We define $\tilde{f}(x) = \lim_{n \rightarrow \infty} f(x_n)$.

- Suppose $x \in S^c$ is not an accumulation point of S . Then there exists an open interval $(a, b) \in S^c$ containing x . Let (l, r) be the largest open interval containing x which is contained in S^c . We allow $l = -\infty$ and $r = \infty$. We consider several cases:
 - If l and r are both real numbers, then l and r are both limit points of S , so $\tilde{f}(l)$ and $\tilde{f}(r)$ can be defined as above. Define $\tilde{f}$ on $[l, r]$

as

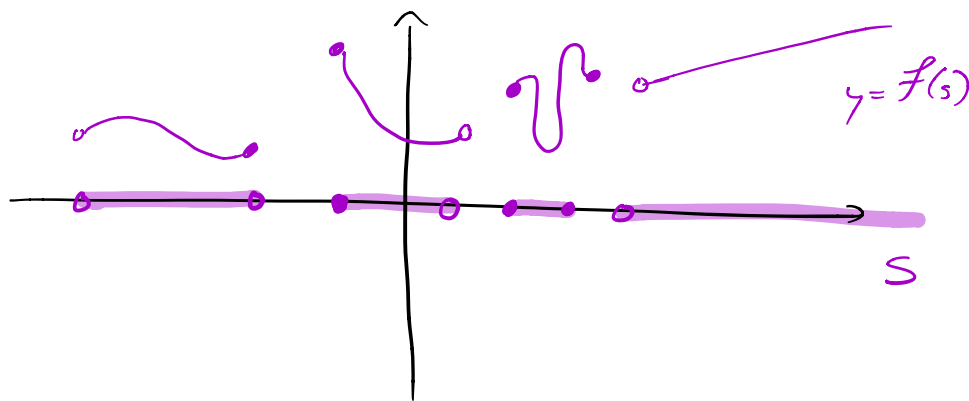
$$\tilde{f}(y) = \frac{y-l}{r-l} \cdot f(r) + \left(1 - \frac{y-l}{r-l}\right) \cdot f(l)$$

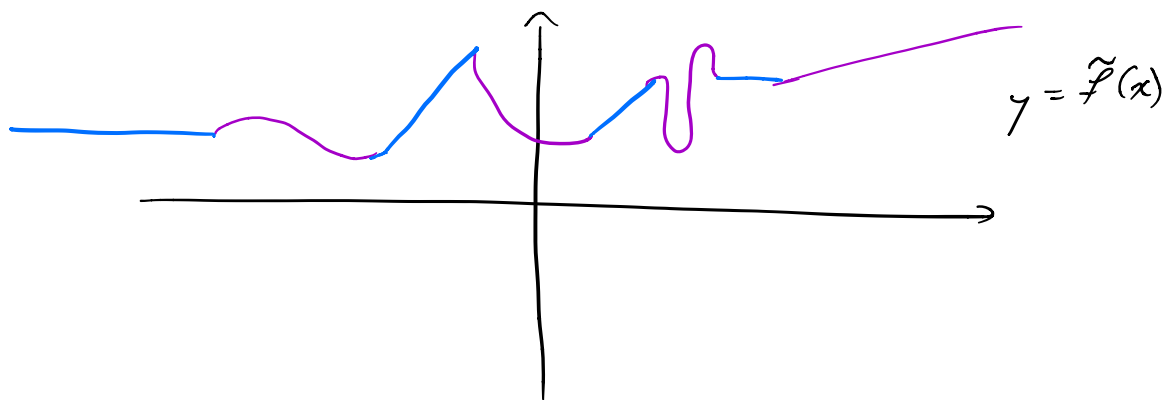
(This is just the function whose graph is the line segment through $(l, f(l))$ and $(r, f(r))$.)

- If $r = \infty$, define $\tilde{f}(x) = \tilde{f}(l)$ for all $x > l$
- If $l = -\infty$, define $\tilde{f}(x) = \tilde{f}(r)$ for all $x < r$.

It remains to show $\tilde{f}$ is continuous, but we'll leave this as an exercise. $\square$

The idea of the proof above is that if f is uniformly continuous on S , "bad things" like vertical asymptotes can't happen, so we just continuously "fill in the gaps" in the graph of f on S .





Now, it can be difficult in general to show a function is uniformly continuous, which is why we like results such as the following:

Thm

Every continuous function on a compact domain is uniformly continuous.

Before proving this it will be helpful to prove one preliminary factoid:

Thm

A function $f: S \rightarrow \mathbb{R}$ is not uniformly continuous iff $\exists \varepsilon > 0$ and a pair of sequences x_n, y_n contained in S such that $|x_n - y_n| \rightarrow 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon$ for all sufficiently large n .

Pf

- Suppose $f: S \rightarrow \mathbb{R}$ is not uniformly continuous,

then $\exists \varepsilon > 0$ such that for all $\delta > 0$ there exists $x, y \in S$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$. Thus there exist x_1, y_1 so that $|x_1 - y_1| < 1$ but $|f(x_1) - f(y_1)| \geq \varepsilon$. There also exist x_2, y_2 so $|x_2 - y_2| < \frac{1}{2}$ but $|f(x_2) - f(y_2)| \geq \varepsilon$; and x_3, y_3 with $|x_3 - y_3| < \frac{1}{3}$, but $|f(x_3) - f(y_3)| \geq \varepsilon$, and so on.

• Now suppose $f: S \rightarrow \mathbb{R}$ has the property that there is some $\varepsilon > 0$ and sequences x_n, y_n in S with $|x_n - y_n| \rightarrow 0$ yet $|f(x_n) - f(y_n)| \geq \varepsilon$. For any $\delta > 0$, $\exists N$ so that $|x_n - y_n| < \delta$ for all $n > N$, yet $|f(x_n) - f(y_n)| \geq \varepsilon$.

□

The above gives us a tool for showing a given function is not uniformly continuous.

Ex

Consider $f: (0, 1) \rightarrow \mathbb{R}$ given by $f(x) = \sin(\frac{1}{x})$.

Consider $\varepsilon = 1$, and

$$x_n = \frac{1}{\frac{\pi}{2} + 2\pi n}$$

$$y_n = \frac{1}{\frac{3\pi}{2} + 2\pi n}$$

Note

$$\begin{aligned} |x_n - y_n| &= \left| \frac{1}{\frac{\pi}{2} + 2\pi n} - \frac{1}{\frac{3\pi}{2} + 2\pi n} \right| \\ &= \left| \frac{2}{\pi + 4\pi n} - \frac{2}{3\pi + 4\pi n} \right| \end{aligned}$$

Then

cl If $f: [a, b] \rightarrow \mathbb{R}$ is cb, it's unif. cb.
pf

Suppose for the sake of contradiction that f was cb, but not unif. cb. Then

$\exists$ seq's $x_n, y_n \in [a, b]$ and an $\varepsilon > 0$ st.

$$|x_n - y_n| \rightarrow 0 \text{ but } |f(x_n) - f(y_n)| > \varepsilon.$$

But x_n has a convergent subseq, x_{n_i} .

Note if $x_{n_i} \rightarrow z$, since

$$y_{n_i} = x_{n_i} + y_{n_i} - x_{n_i}$$

we have $y_{n_i} \rightarrow z$ as well. Thus

$$f(x_{n_i}) - f(y_{n_i}) \rightarrow f(z) - f(z) = 0$$

But $|f(x_{n_i}) - f(y_{n_i})| > \varepsilon. \downarrow$

□

Thm

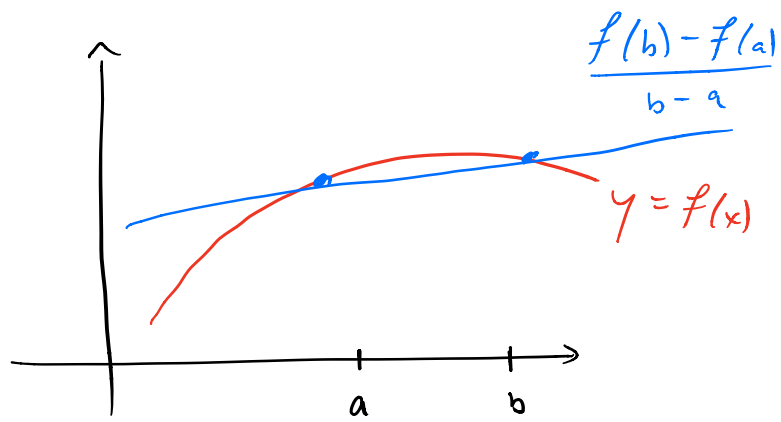
$f: (a, b) \rightarrow \mathbb{R}$ is unif. cts iff
 $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow b^-} f(x)$ exist.

Lecture 13 - Differentiation

Many times it can be hard to work with a mathematical quantity directly, and so we have to work with simpler, easier-to-understand quantities that approximate the value we're truly interested.

In the case of functions of one variable, one class of simple, easy-to-understand functions are linear functions, so we may wish to approximate more complicated functions with linear approximations, and this requires that we understand differentiability and a function's derivatives.

In a first exposure to calculus, derivatives are introduced as the slope of the tangent line to the graph of the function. This "slope" is exactly what we need to define our linear approximation. We can estimate this value by considering difference quotients which give the slope of secant lines approaching the tangent line.



Notice if we set $h = b - a$, then

$$b = b - a + a = a + h$$

and the difference quotient may be written as

$$\frac{f(a+h) - f(a)}{h}$$

We consider the limit as $b \rightarrow a$ (or, equivalently, as $h \rightarrow 0$)

$$\lim_{b \rightarrow a} \frac{f(b) - f(a)}{b - a}$$

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

If this limit exists we say f is differentiable at a and denote the limit $f'(a)$ which is referred to as the derivative of

f at a .

E.g.

Compute $f'(2)$ where $f(x) = 4x^2 + 2x - 1$.

$$f'(2) = \lim_{b \rightarrow 2} \frac{f(b) - f(2)}{b - 2} \quad 4 \cdot 4 + 2 \cdot 2 - 1$$

$$= \lim_{b \rightarrow 2} \frac{4b^2 + 2b - 1 - 19}{b - 2}$$

$$= \lim_{b \rightarrow 2} \frac{4b^2 + 2b - 20}{b - 2}$$

$$= \lim_{b \rightarrow 2} \frac{(b - 2)(4b + 10)}{b - 2}$$

$$= \lim_{b \rightarrow 2} (4b + 10)$$

$$= 18$$

Of course, derivatives need not exist:

If we tried to compute $f'(0)$ where $f(x) = |x|$, we would consider the limit

$$\lim_{h \rightarrow 0} \frac{|0+h| - |0|}{h} = \lim_{h \rightarrow 0} \frac{|h|}{h}$$

Notice, though,

$$\lim_{h \rightarrow 0^+} \frac{|h|}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1$$

$$\lim_{h \rightarrow 0^-} \frac{|h|}{h} = \lim_{h \rightarrow 0} \frac{-h}{h} = -1$$

Thus $\lim_{h \rightarrow 0} \frac{|0+h| - |0|}{h}$ does not exist; we would then say $|x|$ is not differentiable at $x=0$.

As with any limit, we can interpret the derivative in terms of sequences.

Then

f is differentiable at a and $f'(a)=L$ iff for every sequence $x_n \rightarrow a$ (every x_n in domain of f , no x_n equals a), we have

$$\frac{f(x_n) - f(a)}{x_n - a} \rightarrow L$$

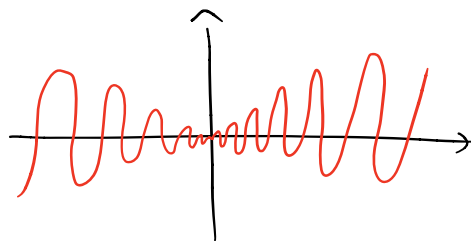
This theorem is an immediate consequence of our "sequential" interpretation of limits of functions.

This can be useful for showing a function is not differentiable.

E.g.

Show the function $f(x)$ below is not differentiable at 0.

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



Consider the sequences

$$x_n = \frac{1}{\frac{\pi}{2} + 2\pi n}$$

$$y_n = \frac{1}{\frac{3\pi}{2} + 2\pi n}$$

Notice

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{f(x_n) - f(0)}{x_n - 0} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{\frac{\pi}{2} + 2\pi n} \cdot \sin\left(\frac{1}{\frac{1}{\frac{\pi}{2} + 2\pi n}}\right)}{\frac{1}{\frac{\pi}{2} + 2\pi n}} \\ &= \lim_{n \rightarrow \infty} \sin\left(\frac{\pi}{2} + 2\pi n\right) = 1 \end{aligned}$$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \frac{f(y_n) - f(0)}{y_n - 0} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{\frac{3\pi}{2} + 2\pi n} \cdot \sin\left(\frac{1}{\frac{3\pi}{2} + 2\pi n}\right)}{\frac{1}{\frac{3\pi}{2} + 2\pi n}} \\
 &= \lim_{n \rightarrow \infty} \sin\left(\frac{3\pi}{2} + 2\pi n\right) \\
 &= -1
 \end{aligned}$$

As these values disagree, the limit $\lim_{b \rightarrow 0} \frac{f(b) - f(0)}{b - 0}$ does not exist, so the function is not differentiable at 0.

If a function f defined on an interval I is differentiable at every point of I , we just say f is differentiable on I . If f is differentiable at every point in its domain, we say f is differentiable.

Provided f is differentiable, we may regard the mapping $x \mapsto f'(x)$ as a function, also called the derivative of f . Ultimately we want to have some tools for computing these $f'(x)$ functions given an expression defining $f(x)$, as well as prove some basic properties of these functions.

Our first basic property is

Prop

If f is differentiable at a , then f is continuous at a .

Pf

We must show $\lim_{x \rightarrow a} f(x) = f(a)$, given that $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$. We simply notice

$$\begin{aligned}\lim_{x \rightarrow a} [f(x) - f(a)] &= \lim_{x \rightarrow a} [f(x) - f(a)] \cdot \frac{x - a}{x - a} \\&= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \cdot (x - a) \\&= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \rightarrow a} (x - a) \\&= f'(a) \cdot 0 \\&= 0\end{aligned}$$

Thus, $\lim_{x \rightarrow a} (f(x) - f(a)) = 0 \Rightarrow \lim_{x \rightarrow a} f(x) = f(a)$. $\square$

It's worth pointing out that while differentiability of f implies continuity of f , the derivative $f'(x)$ need not be continuous.

(An example of such a function will be in the homework.)

In terms of computing derivative, we need a few simple "building blocks" whose derivative we can directly compute, and then some rules for describing how the derivatives of functions built from these building blocks can be computed.

Lemma

- The derivative of a constant function $f(x) = k$ is zero at every point.
- The derivative of the identity function $f(x) = x$ is one at every point.

The proof of the two statements in the lemma will be left as (easy) exercises.

Of course, the constant function and identity function are pretty boring in and of themselves, but we can build up more interesting functions — like polynomials and rational functions — by adding, subtracting, multiplying, and dividing identity and constant functions. Thus, we'd like to know how derivatives "interact" with these operations.

Thm

Suppose $f(x)$ and $g(x)$ are differentiable functions,
then

- $\frac{d}{dx} (f(x) \pm g(x)) = f'(x) \pm g'(x)$
- $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$
- $\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

Pf

- $\lim_{h \rightarrow 0} \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$
 $= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$
 $= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}$
 $= f'(x) + g'(x)$

The proof for $\frac{d}{dx} [f(x) - g(x)] = f'(x) - g'(x)$

is very similar

- $\frac{d}{dx} f(x)g(x) = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$
 $= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$
 $= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \cdot g(x+h) + \lim_{h \rightarrow 0} f(x) \frac{g(x+h) - g(x)}{h}$
 $= f'(x)g(x) + f(x)g'(x)$
- $\frac{d}{dx} \frac{f(x)}{g(x)} = \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$
 $= \lim_{h \rightarrow 0} \frac{g(x) f(x+h) - f(x) g(x+h)}{h g(x+h) g(x)}$

$$= \lim_{h \rightarrow 0} \frac{g(x) f(x+h) - g(x) f(x) + g(x) f(x) - f(x) g(x+h)}{h g(x+h) g(x)}$$

$$= \lim_{h \rightarrow 0} \frac{g(x) [f(x+h) - f(x)] - f(x) [g(x+h) - g(x)]}{h g(x+h) g(x)}$$

$$= \lim_{h \rightarrow 0} \frac{g(x) \cdot \frac{f(x+h) - f(x)}{h} - f(x) \cdot \frac{g(x+h) - g(x)}{h}}{g(x+h) g(x)}$$

$$= \frac{g(x) f'(x) - f(x) g'(x)}{g(x)^2}$$

□

From this we can easily prove the following familiar rules:

Lemma

Suppose $f(x)$ is a differentiable function, k any constant, and $n \geq 1$ is a positive integer. Then

- $\frac{d}{dx} k f(x) = k f'(x)$
- $\frac{d}{dx} x^n = n x^{n-1}$

Pf

$$\begin{aligned} \cdot \frac{d}{dx} k f(x) &= \left[\frac{d}{dx} k \right] f(x) + k \cdot f'(x) = 0 \cdot f(x) + k f'(x) \\ &= k f'(x) \end{aligned}$$

• Here we'll use induction. The base case, $n=1$, is taken care of our earlier lemma

which told us $\frac{d}{dx} x = 1$. That is,

$$\frac{d}{dx} x^1 = 1 \cdot x^0 = 1$$

Now suppose we have proven $\frac{d}{dx} x^{n-1} = (n-1)x^{n-2}$.
We then have

$$\begin{aligned}\frac{d}{dx} x^n &= \frac{d}{dx} x \cdot x^{n-1} \\ &= \left[\frac{d}{dx} x \right] x^{n-1} + x \cdot \frac{d}{dx} x^{n-1} \\ &= x^{n-1} + x \cdot (n-1)x^{n-2} \\ &= x^{n-1} + (n-1)x^{n-1} \\ &= nx^{n-1}\end{aligned}$$

□

So, now we've justified enough that we can differentiate polynomials and rational functions w/ the familiar "tricks" from calculus.
E.g.,

$$\frac{d}{dx} (7x^3 + 5x^2 - 3x + 4) = 21x^2 + 10x - 3$$

$$\frac{d}{dx} \frac{x^3 + x}{x^2 - 1} = \frac{(x^2 - 1) \cdot (3x^2 + 1) - (x^3 + x) \cdot 2x}{(x^2 - 1)^2}$$

There's still plenty we can't differentiate yet,

however. The next big rule we need is the chain rule:

Thm

If f and g are differentiable, then so is their composition $f \circ g$, and

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

Pf

$$\lim_{b \rightarrow x} \frac{f(g(b)) - f(g(x))}{b - x}$$

$$= \lim_{b \rightarrow x} \frac{f(g(b)) - f(g(x))}{b - x} \cdot \frac{g(b) - g(x)}{g(b) - g(x)}$$

$$= \lim_{b \rightarrow x} \frac{f(g(b)) - f(g(x))}{g(b) - g(x)} \cdot \frac{g(b) - g(x)}{b - x}$$

$$= \lim_{b \rightarrow x} \frac{f(g(b)) - f(g(x))}{g(b) - g(x)} \cdot \lim_{b \rightarrow x} \frac{g(b) - g(x)}{b - x}$$

$$= \lim_{b \rightarrow x} \frac{f(g(b)) - f(g(x))}{g(b) - g(x)} \cdot g'(x)$$

Now notice that since g is differentiable, it must be continuous, so $g(b) \rightarrow g(x)$ as $b \rightarrow x$.

Thus the denominator is approaching 0. Let $h = g(b) - g(x)$, and we may write our limit as

$$\lim_{h \rightarrow 0} \frac{f(g(x) + h) - f(g(x))}{h} \cdot g'(x) \\ = f'(g(x)) \cdot g'(x). \quad \square$$

Notice that above we said $\frac{d}{dx} x^n = nx^{n-1}$ if n is a positive integer — our proof would not work for other values for the exponent. However, the chain rule will allow us to apply this rule to some other exponents besides just integers. In particular, by interpreting $\sqrt{x} = x^{1/2}$ as the inverse of x^2 , it suffices to have a rule for differentiating inverse functions.

Then

Suppose f is invertible and differentiable. Then the inverse f^{-1} is also differentiable whenever $f'(f^{-1}(x)) \neq 0$ and satisfies

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Pf Since f and f^{-1} are inverses, we have

$$f(f^{-1}(x)) = x$$

for all x in the domain of f^{-1} . As these two functions are equal, their derivatives are equal, so

$$\frac{d}{dx} f(f^{-1}(x)) = 1$$

By the chain rule this becomes

$$f'(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) = 1$$

$$\Rightarrow \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

□

We can apply this to differentiate $\sqrt{x}$:

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$$

or alternatively

$$\frac{d}{dx} x^{1/2} = \frac{1}{2} x^{-1/2}$$

We may extend this to show for any rational number $\frac{p}{q}$

$$\frac{d}{dx} x^{\frac{p}{q}} = \frac{p}{q} x^{\frac{p}{q}-1}$$

The details will be left as an exercise, but the idea is to write $x^{\frac{p}{q}} = \left(x^{\frac{1}{q}}\right)^p$ and interpret this as a composition and apply the chain rule. When differentiating $x^{\frac{1}{q}}$, treat this as the inverse of x^q and use the inverse derivative rule above.

Let's finish up our discussion of these basic derivative properties by relating the derivative to extrema of a function.

Recall that a point c is a local maximum of a function f defined in an open ball $(c-\varepsilon, c+\varepsilon)$ around c if $f(c) \geq f(x)$ for all x in that open ball - i.e., all $c-\varepsilon < x < c+\varepsilon$.

A local minimum is defined in the same way, but $f(c) \leq f(x)$ for all x in that ε -ball.

Our first result relating local extrema

to differentiability is

Thm (Fermat's Theorem)

If f is a local extremum of f , then either $f'(c) = 0$, or f is not differentiable at c .

Pf
Suppose c is a local minimum: there exists some $\epsilon > 0$ so that $f(c) \leq f(x)$ for all x satisfying $c - \epsilon < x < c + \epsilon$; and suppose also that $f'(c)$ exists. Then we want to show $f'(c)$ must be zero.

By definition,

$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

exists. Let h_n be a sequence approaching 0 with $0 < h_n < \epsilon$ (ϵ the value above).

Then $c + h_n \in (c - \epsilon, c + \epsilon)$ and $f(c + h_n) \geq f(c)$.

Thus

$$\frac{f(c + h_n) - f(c)}{h_n} \geq 0$$

so, $\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \leq 0.$

Now let h'_n be a sequence converging to 0 with $-\varepsilon < h'_n < 0$. Again, $f(c+h'_n) \leq f(c)$, but as $h'_n < 0$, we have

$$\frac{f(c+h'_n) - f(c)}{h'_n} \geq 0$$

So, $\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \geq 0$ as well. Together with our earlier statement this means $f'(c) = 0$. $\square$

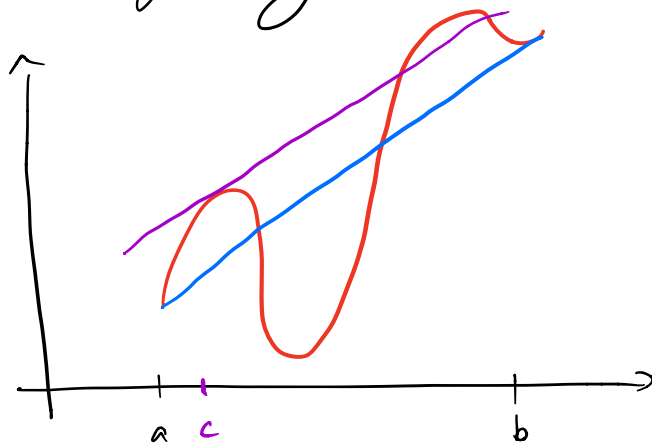
As global extrema are also local extrema, this suggests a method for computing the global extrema of a continuous function $f(x)$ on an interval $[a, b]$: Determine all the points x of $[a, b]$ where $f'(x) = 0$ or is undefined (these are called critical points). By Fermat's theorem, the global extrema must be a critical point or an endpoint a or b of the interval. So, just evaluate $f(x)$ at every critical point and the endpoints and determine which gives the largest and

smallest possible values of $f(x)$.

Lecture 14 - The Mean Value Theorem and Consequences

One of the most important theorems of calculus is something calculus students rarely pay much attention to because its use lies primarily in proving other theorems. As we will see, the mean value theorem is a workhorse behind the scenes in many of calculus' most useful and applied theorems.

Geometrically, the MVT says that if f is differentiable on $[a, b]$, then there exist a point c in (a, b) so that the tangent line to $y = f(x)$ at $x = c$ is parallel to the secant line joining $(a, f(a))$ and $(b, f(b))$.



Before proving the MVT we prove a simpler version called Rolle's theorem.

Lemma (Rolle's Theorem)

Suppose $f: [a, b] \rightarrow \mathbb{R}$ is continuous and is differentiable on (a, b) . If $f(a) = f(b)$, then there exists a c in (a, b) such that $f'(c) = 0$.

Pf

If $f(x) = f(a)$ for all x in $[a, b]$, then it's a constant function and has derivative zero at every point. Otherwise, since f is continuous on a closed, bounded interval so by the extreme value theorem has both a global max and a global min. As $f(a) = f(b)$, at least one of the max or min occurs in (a, b) . Since f is differentiable on (a, b) , Fermat's theorem tells us that the derivative is zero at that extremum. $\square$

Once we have Rolle's theorem at our disposal, proving the mean value theorem becomes very easy.

Thm (Mean Value Theorem)

Suppose f is continuous on $[a, b]$ and differentiable

on (a, b) . Then there exists $c \in (a, b)$ so that
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Pf
Consider the function

$$g(x) = f(x) - \left(\frac{x-a}{b-a} \cdot f(b) + \left(1 - \frac{x-a}{b-a}\right) \cdot f(a) \right)$$

Notice $g(x)$ is continuous on $[a, b]$ and differentiable on (a, b) . In addition,

$$\begin{aligned} g(a) &= f(a) - \left(\frac{a-a}{b-a} \cdot f(b) + \left(1 - \frac{a-a}{b-a}\right) f(a) \right) \\ &= f(a) - (0 + f(a)) \\ &= 0 \end{aligned}$$

$$\begin{aligned} g(b) &= f(b) - \left(\frac{b-a}{b-a} \cdot f(b) + \left(1 - \frac{b-a}{b-a}\right) f(a) \right) \\ &= f(b) - (f(b) + 0) \\ &= 0 \end{aligned}$$

By Rolle's theorem there exists a $c \in (a, b)$ so that $g'(c) = 0$. Notice, though

$$\begin{aligned} g'(x) &= f'(x) - \frac{f(b)}{b-a} + \frac{f(a)}{b-a} \\ &= f'(x) - \frac{f(b) - f(a)}{b-a} \end{aligned}$$

Thus if $g'(c) = 0$, we have

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$
$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

The mean value theorem has some surprising and useful applications.

Ex

Suppose $f(x)$ is defined and differentiable everywhere, and it is known that $f(0) = 2$ and $f'(x) \leq 4$ for all x . What is the largest possible value of $f(10)$?

If we apply the mean value theorem to f on $[0, 10]$, then there exists a c such that

$$f'(c) = \frac{f(10) - f(0)}{10 - 0} = \frac{f(10) - 2}{10}$$

But $f'(x) \leq 4$ for all x , so

$$f'(c) \leq 4$$
$$\Rightarrow \frac{f(10) - 2}{10} \leq 4$$

$$\Rightarrow f(10) \leq 42$$

We can also use the mean value theorem to prove some other useful properties of functions.

Thm

Suppose f is differentiable on (a, b) . Then f is increasing on (a, b) iff $f'(x) \geq 0$ for all $x \in (a, b)$.

pf

Suppose f is increasing. Let x be any point in (a, b) and y_n a sequence where each $x < y_n < b$ and $y_n \rightarrow b$. Consider

$$f'(x) = \lim_{n \rightarrow \infty} \frac{f(y_n) - f(x)}{y_n - x}$$

But the numerator and denominator are both positive for every n , so the limit is positive.

Now suppose $f'(x) \geq 0$ for every point $x \in (a, b)$. Let $a < x < y < b$. We must show $f(x) \leq f(y)$. Now apply the mean value theorem

to f on the interval $[x, y]$. There exists
a c in (x, y) so that

$$f'(c) = \frac{f(y) - f(x)}{y - x}$$

Since $f'(c) \geq 0$, we have

$$0 \leq \frac{f(y) - f(x)}{y - x}$$

$$\Rightarrow 0 \leq f(y) - f(x)$$

$$\Rightarrow f(x) \leq f(y).$$

□

We may similarly prove that for a differentiable function f defined on (a, b) , f is decreasing iff $f'(x) \leq 0$ for all $x \in (a, b)$; and f is constant on (a, b) iff $f'(x) = 0$ for all $x \in (a, b)$.
One corollary of this is

Cor

Suppose f and g are differentiable on (a, b)
and $f'(x) = g'(x)$ for all $x \in (a, b)$. Then
 $f(x) = g(x) + k$ for some constant k .

Pf

Notice $\frac{d}{dx}(f(x)-g(x))=0$ for all x , so $f(x)-g(x)$ is constant. $\square$

There is another, stronger version of the mean value theorem which is sometimes useful, called the Cauchy mean value theorem:

Thm (Cauchy mean value theorem)

Suppose f and g are functions which are continuous on $[a, b]$ and differentiable on (a, b) . Then there exists a $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Consider the function $h(x)$ defined on $[a, b]$ as

$$h(x) = (f(b)-f(a))g(x) - (g(b)-g(a))f(x)$$

$$\begin{aligned} \text{Notice } h(a) &= (f(b)-f(a))g(a) - (g(b)-g(a))f(a) \\ &= f(b)g(a) - f(a)g(a) - f(a)g(b) + f(a)g(a) \end{aligned}$$

$$= f(b)g(a) - f(a)g(b)$$

and

$$\begin{aligned} h(b) &= (f(b) - f(a))g(b) - (g(b) - g(a))f(b) \\ &= f(b)g(b) - f(a)g(b) - f(b)g(b) + f(b)g(a) \\ &= f(b)g(a) - f(a)g(b) \end{aligned}$$

Thus by Rolle's theorem there exists a $c \in (a, b)$ with $h'(c) = 0$. But notice

$$h'(x) = (f(b) - f(a))g'(x) - (g(b) - g(a))f'(x)$$

Thus when $x=c$ we have

$$(f(b) - f(a))g'(c) - (g(b) - g(a))f'(c) = 0$$

$$\Rightarrow (f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$$

$$\Rightarrow \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

□

Notice the normal mean value theorem is a corollary of the Cauchy mean value

Theorem - just take $g(x) = x$.

One example of how the Cauchy mean value theorem may be used is in proving L'Hôpital's rule:

Thm (L'Hôpital's rule)

Suppose $f, g: (a, b) \rightarrow \mathbb{R}$ are continuous and differentiable. If $c \in (a, b)$ with $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$, then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right-hand side exists.

Pf

By the Cauchy mean value theorem, for every $x \in (a, c)$ there's a point $y(x)$ (which depends on x) in (x, c)

$$\frac{f'(y(x))}{g'(y(x))} = \frac{f(c) - f(x)}{g(c) - g(x)}$$

since $x < y(x) < c$

As $x \rightarrow c^-$, $y(x) \rightarrow c^-$ as well and we have

$$\lim_{x \rightarrow c^-} \frac{f'(y(x))}{g'(y(x))} = \lim_{x \rightarrow c^-} \frac{f(c) - f(x)}{g(c) - g(x)}$$

But f and g are continuous and we have assumed $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$. By continuity this means $f(c) = g(c) = 0$. Thus the above becomes

$$\lim_{x \rightarrow c^-} \frac{f'(y(x))}{g'(y(x))} = \lim_{x \rightarrow c^-} \frac{f(x)}{g(x)}$$

Let's notice in general that if $x < k(x) < c$ for all x and if $\lim_{x \rightarrow c^-} l(x)$ exists, then $\lim_{x \rightarrow c^-} l(k(x)) = \lim_{x \rightarrow c^-} l(x)$ (Details left as an exercise.), so the left-hand side above becomes

$$\lim_{x \rightarrow c^-} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow c^-} \frac{f(x)}{g(x)}.$$

Repeating the same argument, applying the Cauchy mean value theorem on the interval (c, x) for $x \in (c, b)$, gives

$$\lim_{x \rightarrow c^+} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow c^+} \frac{f(x)}{g(x)}.$$

And together these tell us

$$\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow c} \frac{f(x)}{g(x)}.$$

□

L'Hôpital's rule is useful for calculating a number of otherwise very difficult limits.

E.g.,

Compute

$$\lim_{x \rightarrow 1} \frac{5x^6 + x^4 - 5x^2 - 1}{9x^7 + x^5 - 10x^4 + 18x^3 + 2x - 20}$$

Notice the numerator and denominator are polynomials, so continuous, and

$$\lim_{x \rightarrow 1} (5x^6 + x^4 - 5x^2 - 1) = 5 + 1 - 5 - 1 = 0$$

$$\lim_{x \rightarrow 1} (9x^7 + x^5 - 10x^4 + 18x^3 + 2x - 20) = 9 + 1 - 10 + 18 + 2 - 20 = 0$$

Thus we may apply L'Hôpital's rule to write

$$\lim_{x \rightarrow 1} \frac{5x^6 + x^4 - 5x^2 - 1}{9x^7 + x^5 - 10x^4 + 18x^3 + 2x - 20} = \lim_{x \rightarrow 1} \frac{30x^5 + 4x^3 - 10x}{63x^6 + 5x^4 - 40x^3 + 54x + 2}$$

$$= \frac{30 + 4 - 10}{63 + 5 - 40 + 54 + 2}$$

$$= \frac{24}{84} = \frac{2}{7}$$

Lecture 15 - The Riemann integral

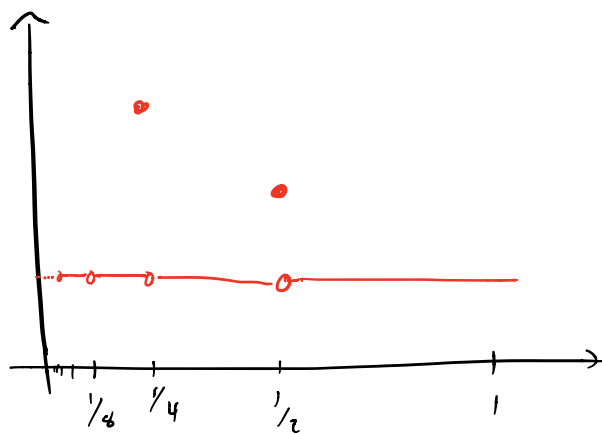
We now turn our attention to what will turn out to be the "opposite" of differentiation: (Riemann) integration.

Though originally motivated by the geometric problem of computing areas, "integration" has become ubiquitous in mathematics. For example, many important and interesting quantities in probability and statistics and physics are defined or computed in terms of integrals. However, several different "types" of integrals exist. Many problems in probability and dynamical systems are concerned with "Lebesgue integration" (Lebesgue is pronounced "luh-bayguh"), which extends the notion of integration you learned in calculus so that a wider class of functions can be integrated, and ^(Lebesgue) integration can also be defined on more abstract spaces besides $\mathbb{R}$ and $\mathbb{R}^n$.

For example, as we'll see soon, Riemann integrals of a function $f: [a, b] \rightarrow \mathbb{R}$ can only be defined if f is bounded. Thus a function such as the following,

$$f: [0, 1] \rightarrow \mathbb{R}$$

$$f(x) = \begin{cases} 2^n & \text{if } x = \frac{1}{2^n}, n \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$



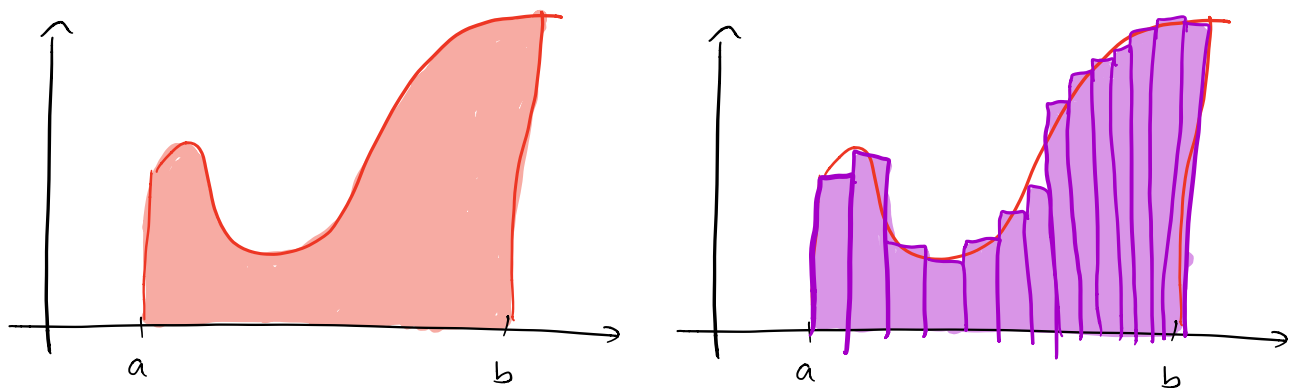
Is not Riemann integrable, however in a Lebesgue integral you're allowed to "ignore" sets of "zero size" which will include any finite or countably infinite sets — such as $\{1/2, 1/4, 1/8, \dots\}$, and in doing so the function becomes Lebesgue integrable with integral 1.

Our goal in this class will be to study the Riemann integral, but we mention the above to point out there are other integration theories, so the adjective "Riemann" is not redundant. Having said this, we'll sometimes just say "integral" when we mean "Riemann

integral," since this is the only integration theory we will consider.

The motivating problem for integration is determining the area between a graph $y = f(x)$ and an interval $[a, b]$ in the domain of f , where for the moment we'll suppose $f(x) > 0$ for all $x \in [a, b]$.

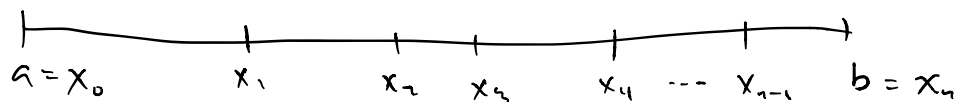
The idea will be to approximate this area with an easier-to-compute quantity: sums of areas of rectangles which approximate our region.



To be precise, we introduce the idea of a "tagged partition." Recall that a partition of $[a, b]$ is a set $P = \{x_0, x_1, x_2, \dots, x_n\}$ where

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

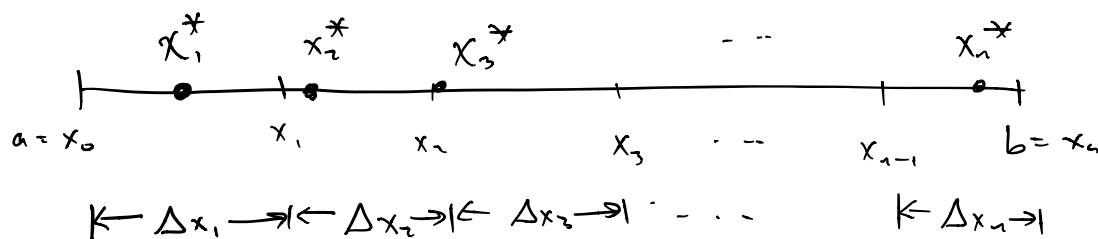
So, P allows us to break $[a, b]$ into subintervals.



We will let Δx_i denote the length of the i^{th} subinterval, $\Delta x_i = x_i - x_{i-1}$. We will eventually care about the length of the largest subinterval and call this the norm of P , denoted $|P|$:

$$|P| := \max \{ \Delta x_1, \Delta x_2, \dots, \Delta x_n \}$$

A tagged partition P^* is a pair $P^* = (P, T)$ where T is a set of "tags" - points $x_i^* \in [x_{i-1}, x_i]$.

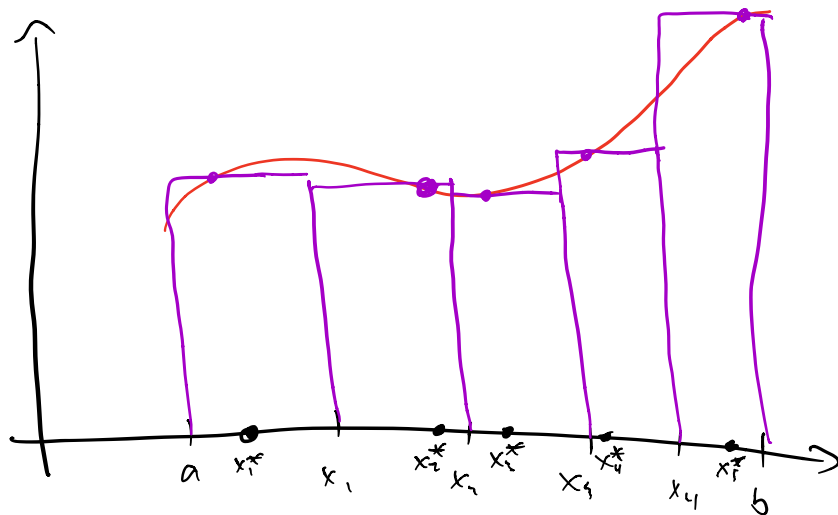


Given a function f defined on $[a, b]$ and

^ a tagged partition P^* , we define the Riemann sum of f wrt P^* as

$$S(f, P^*) = \sum_{i=1}^n f(x_i^*) \Delta x_i$$

In terms of our "areas of rectangles" picture, $f(x_i^*)$ is the height of the i^{th} rectangle, Δx_i is the width, and so $f(x_i^*) \Delta x_i$ is its area



Intuitively we want to take the limit of Riemann sums $S(f, P^*)$ as the widths of these rectangles (i.e., the norms $|P|$) go to zero. We will make this precise as follows.

A function f defined on (a,b) is (Riemann) integrable if there exists an $L \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists $\delta > 0$ so that for all tagged partitions P^* satisfying $|P^*| < \delta$ we have $|S(f, P^*) - L| < \epsilon$. This L , if it exists, is called the Riemann integral of f over $[a,b]$ and denoted $\int_a^b f(x) dx$.

The definition above can be tedious to use, so it's helpful to have some equivalent criteria.

First, let's suppose $f: [a,b] \rightarrow \mathbb{R}$ is bounded. Then for any given partition P of $[a,b]$ let's define two quantities we'll denote $U(f, P)$ and $L(f, P)$ as follows. First, for each subinterval $[x_{i-1}, x_i]$ determined by P , define

$$m_i = \inf \{ f(x) \mid x \in [x_{i-1}, x_i] \}$$

$$M_i = \sup \{ f(x) \mid x \in [x_{i-1}, x_i] \}$$

We then define

$$L(f, P) = \sum_{i=1}^n m_i \Delta x_i$$

$$U(f, P) = \sum_{i=1}^n M_i \Delta x_i$$

Lemma

For every tagged partition P^* ,

$$L(f, P) \leq S(f, P^*) \leq U(f, P)$$

(Here P is P^* sans tags.)

Pf

Just observe for every $x_i^* \in [x_{i-1}, x_i]$,

$$m_i \leq f(x_i^*) \leq M_i$$

□

Now given any two partitions P_1, P_2 of $[a, b]$, we'll say P_2 is a refinement of P_1 if $P_2 \supseteq P_1$. (i.e., every subinterval determined by P_1 may be broken up into more subintervals by P_2).



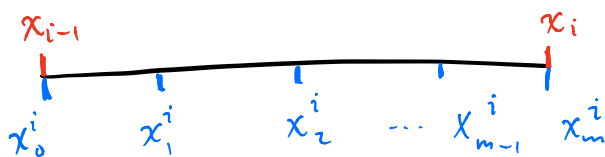
P_1
 P_2

Lemma

If $P_2 \supseteq P_1$ is a refinement of partitions,
then $U(P_2, f) \leq U(P_1, f)$ and $L(P_2, f) \geq L(P_1, f)$.

Pf

Suppose $[x_{i-1}, x_i]$ is a subinterval determined by P_1 and that P_2 breaks this up into $[x_0^i, x_1^i], [x_1^i, x_2^i], \dots, [x_{m-1}^i, x_m^i]$:



Notice $\sup \{f(x) \mid x \in [x_{j-1}^i, x_j^i]\} \leq \sup \{f(x) \mid x \in [x_{i-1}, x_i]\}$
 M_j^i " M_i

Our corresponding sums have

$$\begin{aligned} \sum_{j=1}^m M_j^i (x_{j-1}^i - x_j^i) &\leq \sum_{j=1}^m M_i (x_{j-1}^i - x_j^i) \\ &= M_i \sum_{j=1}^m (x_{j-1}^i - x_j^i) \\ &= M_i (x_{i-1} - x_i). \end{aligned}$$

Applying this to every subinterval of

P_i gives $U(f, P_i) \leq U(f, P_1)$. The result for L 's is similar. $\square$

Lemma

For any two partitions P, P' (not necessarily refinements of one another),

$$L(f, P) \leq U(f, P')$$

Pf

Note P and P' are subsets of $P \cup P'$.

Hence

$$U(f, P \cup P') \leq U(f, P')$$

$$L(f, P \cup P') \geq L(f, P).$$

Thus

$$L(f, P) \leq \underbrace{L(f, P \cup P')}_{\text{From earlier lemma}} \leq U(f, P \cup P') \leq U(f, P). \quad \square$$

Now let Π be the set of all partitions P of $[a, b]$. We define

$$U(f) = \inf \{ U(f, P) \mid P \in \Pi \}$$

$$L(f) = \sup \{ L(f, P) \mid P \in \Pi \}$$

Lemma

$U(f) = L(f)$ iff for every $\varepsilon > 0$ there exists a partition P_ε such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

Pf

• Suppose $U(f) = L(f)$. Then for every $\varepsilon > 0$ there exists partitions P and P' such that

$$U(f, P) < U(f) + \varepsilon/2$$

$$L(f, P') > L(f) - \varepsilon/2$$

Define $P_\varepsilon = P \cup P'$. Then $U(f, P_\varepsilon) < U(f, P)$
and $L(f, P_\varepsilon) > L(f, P')$. Hence

$$U(f, P_\varepsilon) < U(f) + \varepsilon/2$$

$$L(f, P_\varepsilon) > L(f) - \varepsilon/2$$

Multiplying the second inequality by -1 , we have

$$U(f, P_\varepsilon) < U(f) + \varepsilon/2$$

$$-L(f, P_\varepsilon) < \varepsilon/2 - L(f)$$

Adding these inequalities gives

$$U(f, P_\epsilon) - L(f, P_\epsilon) < U(f) - L(f) + \epsilon$$

But $U(f) = L(f)$, and we have the desired inequality.

- Suppose $\exists P_\epsilon$ st. $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$.
Ob $U(f) \leq U(f, P_\epsilon)$ and $L(f) \geq L(f, P_\epsilon)$ (so, $-L(f) \leq -L(f, P_\epsilon)$) we have

$$U(f) - L(f) \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$$

Letting $\epsilon \rightarrow 0$ gives $U(f) - L(f) \leq 0$. But $U(f) \geq L(f)$, so $U(f) - L(f) \geq 0$. Thus $U(f) - L(f) = 0$ and $U(f) = L(f)$. □

Lemma

For every $\epsilon > 0$ and every partition P , $\exists$ tags for P st. $U(f, P) - S(f, P^*) < \epsilon$, and similarly a (possibly different) set of tags so $S(f, P^*) - L(f, P) < \epsilon$.

Pf
Exercise

Thm (Cauchy criterion for Riemann integrability)
 $f: [a, b] \rightarrow \mathbb{R}$ is Riemann integrable iff for every $\varepsilon > 0$ there is a $\Delta > 0$ such that for all tagged partitions P_1^*, P_2^* such that $|P_i^*| < \Delta$, $|S(f, P_1^*) - S(f, P_2^*)| < \varepsilon$.

Pf
 Suppose f is Riemann integrable: $\exists L \in \mathbb{R}$ st.
 $\forall \varepsilon > 0 \exists \delta > 0$ so that if $|P| < \delta$, then
 $|S(f, P^*) - L| < \varepsilon$. Let Δ be st. $|P| < \Delta \Rightarrow |S(f, P^*) - L| < \varepsilon/2$.
 Then

$$\begin{aligned} & |S(f, P_1^*) - S(f, P_2^*)| \\ &= |S(f, P_1^*) - L + L - S(f, P_2^*)| \\ &\leq |S(f, P_1^*) - L| + |S(f, P_2^*) - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

For the converse, see pg. 172 of Gordon. $\square$

Thm

f is Riemann integrable iff $U(f) = L(f)$.

Pf

• Suppose f is Riemann integrable. Then there exists an L so that ^{$\forall \epsilon > 0 \exists \delta > 0$ s.t.} for every partition P^* satisfying $|P| < \delta$, $|S(f, P^*) - L| < \epsilon/2$.

We also have $L(f, P) \leq S(f, P^*) \leq U(f, P)$

By changing tags if necessary we also have $U(f, P) - S(f, P^*) < \epsilon/2$. Thus

$$\begin{aligned} |U(f, P) - L| &\leq |U(f, P) - S(f, P^*)| + |S(f, P^*) - L| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

Similarly, $|L(f, P) - L| < \epsilon$. Hence

$$U(f, P) - L(f, P)$$

can be made less than 2ϵ . But

$U(f, P) \geq U(f)$ and $L(f, P) \leq L(f)$, so

$$U(f) - L(f) < 2\epsilon$$

Letting ϵ go to zero gives $U(f) = L(f)$.

Suppose $U(f) = L(f)$. Let $\varepsilon > 0$ be given. $\exists P_\varepsilon$ st.

$$L(f) - L(f, P_\varepsilon) < \varepsilon.$$

$$M = \sup \{f(x) \mid x \in [a, b]\}$$

$$m = \inf \{f(x) \mid x \in [a, b]\}$$

Let $\delta = \frac{1}{2} \min \{ \Delta x_1, \dots, \Delta x_n, n(2M - m) \}$
where these are subintervals of P_ε .

Let P be any partition st. $|P| < \delta$.

Notice every interval of P contains at most one pt of P_ε (P is "almost" a refinement of P)

Let $P' = P \cup P_\varepsilon$ be the common refinement of P and P_ε . Notice $P' = P \cup \{x_1, x_2, \dots, x_n\}$.

Say $y_1, z_1, y_2, z_2, \dots, y_n, z_n \in P'$ st. $y_i < x_i < z_i$
and n^{th} elt of P' is in $[y_i, z_i]$. Then

$$U(f, P') - U(f, P) =$$

$$\sum_{i=1}^n [M_i(x_i - y_i) + N_i(z_i - x_i) - Q_i(z_i - y_i)]$$

$$\sup \{f(x) \mid x \in [y_i, z_i]\}$$

$$\inf \{f(x) \mid x \in [y_i, x_i]\}$$

$$\sup \{f(x) \mid x \in [x_i, z_i]\}$$

$$\leq \sum_{i=1}^n [M(x_i - y_i) + M(z_i - x_i) - m(z_i - y_i)]$$

$$\leq M \sum_{i=1}^n [(x_i - y_i) + (z_i - x_i)] - m \sum_{i=1}^n (z_i - y_i)$$

$$\leq M_n(2\delta) - m_n\delta$$

$$= n\delta(2M - m)$$

$$< n \frac{\varepsilon}{n(2M - m)} (2M - m)$$

$$= \varepsilon.$$

D

Lemma

If $f: [a, b] \rightarrow \mathbb{R}$ is Riemann integrable, then f must be bounded.

Pf

Suppose instead f was unbounded. Let $\varepsilon > 0$ be given. If f was integrable with $\int_a^b f(x) dx = L$, then there would exist a $\delta > 0$ such that $|P| < \delta \Rightarrow |S(f, P^*) - L| < \varepsilon$ regardless of the tags used. Since f is unbounded, $\exists x_1, x_2, x_3, \dots$ in $[a, b]$ with $|f(x_n)| > N$. For any $\delta > 0$, though, we may consider a partition P with $|P| < \delta$ and one of the chosen tags x_i^* satisfying $|f(x_i^*)| > (|L| + 2\varepsilon) \cdot \frac{1}{|P|}$

$$|S(f, P^*) - L| = \left| \sum f(x_i^*) \Delta x_i - L \right|$$

$$\geq \left| \left| \sum f(x_i^*) \Delta x_i \right| - |L| \right|$$

$$\geq \left| \sum |f(x_i^*)| \Delta x_i - |L| \right|$$

$$\geq \left| |f(x_{i_0}^*)| \cdot \Delta x_i - |L| \right|$$

$$\geq \left| (|L| + 2\varepsilon) \cdot \frac{1}{|P|} \cdot \Delta x_i - |L| \right|$$

$$\geq \left| |L| + 2\varepsilon - |L| \right|$$

$$= 2\varepsilon$$

$$> \varepsilon.$$

□

Lemma

If f is Riemann integrable, then

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$

Pf Exercise

Thm (Fundamental Theorem of Calculus)

Suppose $f: [a, b] \rightarrow \mathbb{R}$ is Riemann integrable.

Define a function $F: [a, b] \rightarrow \mathbb{R}$ by

$$F(x) = \int_a^x f(t) dt$$

Then F is continuous on $[a, b]$, differentiable where f is continuous, and $F'(x) = f(x)$.

Pf

Since f is integrable, it is bounded.

Suppose $|f(x)| \leq M$ for all $x \in [a, b]$.

Let $x, y \in [a, b]$ be any points - say $x < y$.

Note

$$\left| F(y) - F(x) \right| = \left| \int_a^y f(t) dt - \int_a^x f(t) dt \right|$$

$$= \left| \int_x^y f(t) dt \right|$$

$$\leq \int_x^y |f(t)| dt$$

$$\leq M |y - x|$$

Thus F is Lipschitz continuous and hence unif. cb on $[a, b]$.

Suppose f is continuous at $x_0 \in (a, b)$.

$$F'(x) = \lim_{x \rightarrow x_0} \frac{F(x) - F(x_0)}{x - x_0}$$

$$= \lim_{x \rightarrow x_0} \frac{1}{x - x_0} \left[\int_a^x f(t) dt - \int_a^{x_0} f(t) dt \right]$$

$$= \lim_{x \rightarrow x_0} \frac{1}{x-x_0} \int_{x_0}^x f(t) dt$$

We WTS $F'(x_0) = f(x_0)$, which by the above is equivalent to

$$\lim_{x \rightarrow x_0} \int_{x_0}^x f(t) dt = f(x_0).$$

We'll use the ϵ - δ definition of the limit: given any $\epsilon > 0$ we must exhibit a $\delta > 0$ such that if $0 < |x - x_0| < \delta$,

$$\left| \frac{1}{x-x_0} \int_{x_0}^x f(t) dt - f(x_0) \right| < \epsilon.$$

Notice, though,

$$f(x_0) = \frac{1}{x-x_0} \int_{x_0}^x f(x_0) dt$$

Thus our expression above becomes

$$\left| \frac{1}{x-x_0} \int_{x_0}^x (f(t) - f(x_0)) dt \right| < \epsilon$$

Since f is continuous at x_0 , $\exists \delta$ such that $0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$. Thus

$$\left| \frac{1}{x-x_0} \int_{x_0}^x (f(t) - f(x_0)) dt \right|$$

$$\leq \frac{1}{|x-x_0|} \int_{x_0}^x |f(t) - f(x_0)| dt$$

$$\leq \frac{1}{|x-x_0|} \int_{x_0}^x \varepsilon dt$$

$$= \frac{1}{|x-x_0|} \cdot \varepsilon |x-x_0|$$

$$= \varepsilon.$$

□

Cor

If f has antiderivative F (i.e., $F'(x) = f(x)$)
then

$$\int_a^b f(x) dx = F(b) - F(a).$$

□